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NUMERICAL CRITERIA FOR VERY AMPLENESS OF DIVISORS ON PROJECTIVE BUNDLES OVER AN ELLIPTIC CURVE

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ABSTRACT. Let D be a divisor on a projectivized bundle over an elliptic curve. Numerical conditions for the very ampleness of D are proved. In some cases a complete numerical characterization is found.

1. **Introduction.** Ampleness of divisors on algebraic varieties is a numerical property. On the other hand it is in general very difficult to give numerical necessary and sufficient conditions for the very ampleness of divisors. In [4] the author gives a sufficient condition for a line bundle associated with a divisor D to be normally generated on $X = \mathbb{P}(E)$ where E is a vector bundle over a smooth curve C. A line bundle which is ample and normally generated is automatically very ample. Therefore the condition found in [4], together with Miyaoka's well known ampleness criterion, gives a sufficient condition for the very ampleness of D on X. This work is devoted to the study of numerical criteria for very ampleness of divisors D which do not satisfy the above criterion, in the case of C elliptic. With this assumption Biancofiore and Livorni [3] (see also [2, Proposition 8.5.8] for a generalization) gave a necessary and sufficient condition when E is indecomposable, $\operatorname{rk} E = 2$ and $\deg E = 1$. Gushel [6] also gave a complete characterization of the very ampleness of D assuming that E is indecomposable and $|D|$ embeds X as a scroll. This work deals with the general situation and addresses the cases still open.

The main technique used here is a very classical one. A suitable divisor A on X is chosen such that there exists a smooth $S \in |A|$ containing every pair of points, possibly infinitely near. Appropriate vanishing conditions are established to assure that the natural restriction map $H^0\big(X, O_X(D)\big) \longrightarrow H^0\big(S, O_X(D)|_S\big)$ is surjective. In this way we get that a divisor D of X is very ample if and only if $D|_S$ is very ample. In this context S is chosen as $S = \mathbb{P}(E')$ where E' is a quotient of E, thus with rank smaller than rank E. Therefore an inductive process on the rank can be set up. This process is not always easy to carry on. For example if E is assumed to be indecomposable there is no guarantee that E' will still be indecomposable. Since ampleness is inherited by quotients, we will require at some stage that E be ample. The paper is organized as follows. Section 2 contains notation, known and preliminary results used in the sequel. In Section 3 the case of rank $E = 2$ is

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fully treated. We recover Biancofiore and Livorni's results and deal with the case of E decomposable. Section 4 deals with the case of rank $E = 3$ while Section 5 contains the study of case rank $E \geq 4$. In particular in the case of rank $E = 3$ we get the following result (see Section 2.1 for notation):

THEOREM . *Let E be a rank 3 vector bundle on an elliptic curve C and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$.*

- *(a) If E is indecomposable then*
  - *(a1) if $d = 0 \pmod 3$, D is very ample if and only if $b + a\mu^-(E) \geq 3$,*
  - *(a2) if $d = 1 \pmod 3$, D is very ample if $b + a\mu^-(E) > 1$,*
  - *(a3) if $d = 2 \pmod 3$, D is very ample if $b + a\mu^-(E) > \frac{4}{3}$,*
- *(b) if E is decomposable, then D is very ample if and only if $b + a\mu^-(E) \geq 3$ except when $E = E_1 \oplus E_2$, with $\operatorname{rk} E_1 = 1$, $\operatorname{rk} E_2 = 2$, $\deg E_2$ odd and $\deg E_1 > \frac{\deg E_2}{2}$. In the latter case the condition is only sufficient.*

Notice that the above theorem shows the existence, among others, of a smooth threefold of degree 20 embedded in $\mathbb{P}^9$ as a fibration of Veronese surfaces over an elliptic curve, choosing $a = 2$, $b = -1$ and $d = 4$.

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2. General results and preliminaries.

2.1. *Notation.* The notation used in this work is mostly standard from Algebraic Geometry. Good references are [9] and [5]. The ground field is always the field $\mathbb{C}$ of complex numbers. Unless otherwise stated all varieties are supposed to be projective. $\mathbb{P}^n$ denotes the n-dimensional complex projective space and $\mathbb{C}^*$ the multiplicative group of non zero complex numbers. Given a projective n-dimensional variety X, $\mathcal{O}_X$ denotes its structure sheaf and $\operatorname{Pic}(X)$ denotes the group of line bundles over X. Line bundles, vector bundles and Cartier divisors are denoted by capital letters as $L, M, \ldots$. Locally free sheaves of rank one, line bundles and Cartier divisors are used interchangeably as customary. Let $L, M \in \operatorname{Pic}(X)$, let E be a vector bundle of rank r on X, let $\mathcal{F}$ be a coherent sheaf on X and let $Y \subset X$ be a subvariety of X. Then the following notation is used:

LC	the intersection number of L with a curve C,
L^n	the degree of L,
$\lvert L\rvert$	the complete linear system of effective divisors associated with L,
$L\lvert_Y$	the restriction of L to Y,
$L \sim M$	the linear equivalence of divisors,
$L \equiv M$	the numerical equivalence of divisors,
$\operatorname{Num}(X)$	the group of line bundles on X modulo the numerical equivalence,
E^*	the dual of E,
$\mathbb{P}(E)$	the projectivized bundle of E,

$H^i(X, \mathcal{F})$ the i-th cohomology vector space with coefficient in $\mathcal{F}$,
$h^i(X, \mathcal{F})$ the dimension of $H^i(X, \mathcal{F})$.

If C denotes a smooth projective curve of genus g, and E a vector bundle over C of $\deg E = c_1(E) = d$ and $\operatorname{rk} E = r$, we need the following standard definitions:

E is *normalized* if $h^0(E) \neq 0$ and $h^0(E \otimes L) = 0$ for any invertible sheaf L over C with $\deg L < 0$.

E has slope $\mu(E) = \frac{d}{r}$.

E is *semistable* if and only if for every proper subbundle S, $\mu(S) \leq \mu(E)$. It is *stable* if and only if the equality is strict.

The Harder-Narasimhan filtration of E is the unique filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$$

such that $\frac{E_i}{E_{i-1}}$ is semistable for all i, and $\mu_i(E) = \mu(\frac{E_i}{E_{i-1}})$ is a strictly decreasing function of i.

We recall now some definitions from [4] which we will use in the following: let $0 = E_0 \subset E_1 \subset \cdots \subset E_s = E$ be the Harder-Narasimhan filtration of a vector bundle E over C. Then

$\mu^-(E) = \mu_s(E) = \mu(\frac{E_s}{E_{s-1}})$,
$\mu^+(E) = \mu_1(E) = \mu(E_1)$,
or alternatively
$\mu^+(E) = \max\{\mu(S) \mid 0 \longrightarrow S \longrightarrow E\}$,
$\mu^-(E) = \min\{\mu(Q) \mid E \longrightarrow Q \longrightarrow 0\}$.

We have also $\mu^+(E) \geq \mu(E) \geq \mu^-(E)$ with equality if and only if E is semistable. In particular if C is an elliptic curve, an indecomposable vector bundle E on C is semistable and hence $\mu(E) = \mu^-(E)$. Moreover if F, G are indecomposable and hence semistable vector bundles on an elliptic curve C and $F \longrightarrow G$ is a non zero map, it follows that $\mu(F) \leq \mu(G)$.

2.2. *General results.* Let C be a smooth projective curve of genus g, E a vector bundle of rank r, with $r \geq 2$, over C and $\pi: X = \mathbb{P}(E) \longrightarrow C$ the projective bundle associated to E with the natural projection π. With standard notations denote with $\mathcal{T} = \mathcal{O}_{\mathbb{P}(E)}(1)$ the tautological sheaf and with $\mathcal{F}_P = \pi^* \mathcal{O}_C(P)$ the line bundle associated with the fiber over $P \in C$. Let T and f denote the numerical classes respectively of $\mathcal{T}$ and $\mathcal{F}_P$.

Let $D \sim a\mathcal{T} + \pi^* B$, with $a \in \mathbb{Z}$, $B \in \operatorname{Pic}(C)$ and $\deg B = b$, then $D \equiv aT + bf$. Moreover $\pi_* D = S^a(E) \otimes \mathcal{O}_C(B)$ and hence $\mu^-(\pi_* D) = a\mu^-(E) + b$ (see [4]).

Regarding the ampleness, the global generation, and the normal generation of D, the following criteria are known:

THEOREM 2.1 (MIYAOKA [10]). *Let E be a vector bundle over a smooth projective curve C of genus g, and $X = \mathbb{P}(E)$. If $D \equiv aT + bf$ is a line bundle over X, then D is ample if and only if $a > 0$ and $b + a\mu^-(E) > 0$.*

PROPOSITION 2.2 (GUSHEL [7], PROPOSITION 3.3). *Let $D \sim a\mathcal{T} + \pi^* B$ where $a > 0$ and $B \in$ Pic(C), be a divisor on a projective bundle $\pi: X = \mathbb{P}(E) \longrightarrow C$. Then:*

i) if $a = 1$, the bundle $\pi_*(D)$ is generated by global sections if and only if the divisor D is,

ii) if $a \geq 2$, and the vector bundle $\pi_*(D)$ is generated by global sections, then also the divisor D is.

LEMMA 2.3 (GUSHEL [6], PROPOSITION 3.2). *Let E be an indecomposable vector bundle over an elliptic curve C. E is globally generated if and only if* $\deg E > \operatorname{rank} E$.

LEMMA 2.4 (SEE *e.g.* [4], LEMMA 1.12). *Let E be a vector bundle over C of genus g.*

i) if $\mu^-(E) > 2g-2$ then $h^1(C,E) = 0$,

ii) if $\mu^-(E) > 2g-1$ then E is generated by global sections.

For the following theorem we need a definition:

DEFINITION 1 (BUTLER [4]). Let E be a vector bundle over a variety Y, and let $\pi: X = \mathbb{P}(E) \longrightarrow Y$ be the natural projection. A coherent sheaf $\mathcal{F}$ over X is said to be *$t\pi$-regular* if, for all $i > 0$,

$$\mathcal{R}^i\pi_*\big(\mathcal{F}(t-i)\big) = 0.$$

THEOREM 2.5 (BUTLER [4]). *Let E be a vector bundle over a smooth projective curve C of genus g, and $X = \mathbb{P}(E)$. If D is a $(-1)\pi$-regular line bundle over X, with $\mu^-(\pi_*D) > 2g$, then D is normally generated.*

REMARK 2.6. Let D be a divisor of $X = \mathbb{P}(E)$, with E vector bundle on a smooth projective curve of genus g. As $h^i\big(\mathcal{F}_P, D_{|\mathcal{F}_P}(-1-i)\big) = 0$ for $i \geq 1$, the $(-1)\pi$-regularity of D is satisfied, hence the condition $a\mu^-(E) + b > 2g$ implies that D is normally generated. If a line bundle D on a projective variety X is ample and normally generated it is very ample. Hence from Theorem 2.1 and 2.5 we get that D is very ample on $X = \mathbb{P}(E)$ if

$$b + a\mu^-(E) > 2g. \tag{1}$$

Hence, if $g = 1$, the very ampleness of $D \equiv aT + bf$ is an open problem only in the range

$$0 < b + a\mu^-(E) \leq 2. \tag{2}$$

2.3. *Preliminaries.* The following result is standard from the theory of vector bundles (see [9]):

LEMMA 2.7. *Let E be an indecomposable vector bundle of* rank *r on an elliptic curve. If E is normalized then* $0 \leq \deg E \leq r-1$.

LEMMA 2.8. *Let $E = \bigoplus_{i=1}^n E_i$ be a decomposable vector bundle over an elliptic curve C, with E_i indecomposable vector bundles. Then* $\mu^-(E) = \min \mu(E_i)$.

PROOF. For the proof we need the following three claims.

CLAIM 1. *Let $E = \bigoplus_i E_i$ be as above, then* $\mu(E) \geq \min \mu(E_i)$.

PROOF. Let us denote by $r = \mathrm{rk}(E)$, $r_i = \mathrm{rk}(E_i)$, $d = \deg(E)$, $d_i = \deg(E_i)$. Let us consider the vectors $\underline{v}_i$ in $\mathbb{R}^2$ whose coordinates are (r_i, d_i) and the vector $\underline{v} = \sum_i \underline{v}_i$. Let α_i be the angle between the r-axis and $\underline{v}_i$. Let α be the angle between the r-axis and $\underline{v}$. It is $\mu(E) = \frac{d}{r} = \mathrm{tg}(\alpha) \geq \min_i \mathrm{tg}(\alpha_i) = \min_i(\frac{d_i}{r_i}) = \min_i \mu(E_i)$. ■

CLAIM 2. *Let* $E = \oplus_i E_i$ *be as above, and* $\mu(E_i) = \frac{d_i}{r_i} = h \in \mathbb{Q}$, *for all* i. *Then* $\mu^-(E) = h$.

PROOF. Notice that under this hypothesis $\mu(E) = h$. Moreover, by definition, it is $\mu^-(E) = \min\{\mu(Q) \mid E \longrightarrow Q \longrightarrow 0\}$. If Q is decomposable in the direct sum of indecomposable vector bundles Q_k, the existence of a surjective map $E \longrightarrow Q \longrightarrow 0$ implies the existence of surjective maps $E \longrightarrow Q_k \longrightarrow 0$ for all k and consequently from Claim 1, $\mu^-(E) = \min\{\mu(Q) \mid E \longrightarrow Q \longrightarrow 0$, and Q indecomposable$\}$.

Now let Q_o be an indecomposable vector bundle which realizes the minimum, *i.e.* $\mu(Q_o) = \mu^-(E)$. From $\oplus_i E_i \longrightarrow Q_o \longrightarrow 0$ it follows that there exists at least an index i_0 such that the map $E_{i_0} \longrightarrow Q$ is not zero and $\mu(E_{i_0}) \leq \mu(Q_o)$. Therefore it is $h \leq \mu^-(E)$. As $h \geq \mu^-(E)$, the claim is proved. ■

CLAIM 3. *Let* $E = \oplus_i E_i$ *be as in Claim (2). Then* E *is semistable.*

PROOF. It is enough to prove that for any S vector bundle on C such that there exists a map $0 \longrightarrow S \longrightarrow E$ then $\mu(S) \leq \mu(E) = h$. If we consider the dual map $E^* \longrightarrow S^* \longrightarrow 0$ we have $\mu(S^*) \geq \mu^-(E^*) = \mu^-(\oplus_i E_i{}^*)$ and, as $\mu(E^*{}_i) = -\frac{d_i}{r_i} = -h$, from Claim 2 applied to E^* we have $\mu^-(E^*) = -h$. Hence $\mu(S^*) = -\mu(S) \geq -h$ and $\mu(S) \leq h$. ■

The lemma can now be proved.

Let $E = \oplus_i E_i$ be as in the hypothesis of the lemma, and denote by $\mu_i = \mu(E_i)$. We can choose an ordering such that $E = E_1 \oplus E_2 \oplus E_3 \cdots$ and $\mu_1 \geq \mu_2 \geq \mu_3 \cdots$. Let $E = \oplus_{k=1}^s A_k$ be a new decomposition of E such that each A_k is an indecomposable vector bundle or a sum of indecomposable vector bundles E_i with the same μ_i. In this way we get a strictly decreasing sequence $\mu(A_1) > \mu(A_2) > \cdots > \mu(A_s)$, and by claim (3) each A_k is semistable. Moreover the sequence $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$ with $F_i = A_1 \oplus A_2 \oplus \cdots \oplus A_i$ with $1 \leq i \leq s$, is the Harder-Narasimhan filtration of E because the sequence of the slopes $\mu(\frac{F_i}{F_{i-1}}) = \mu(A_i)$ is strictly decreasing and each $\frac{F_i}{F_{i-1}} = A_i$ is semistable for all $i = 1 \cdots s$. Hence we get $\mu^-(E) = \mu(\frac{E_s}{E_{s-1}}) = \mu(A_s) = \min \mu(E_i)$. ■

LEMMA 2.9. *Let* $D \sim a\mathcal{T} + \pi^*B$ *be a line bundle in* $X = \mathbb{P}(E)$ *over a curve* C *of genus* $g = 1$, *with* $B \in \mathrm{Pic}(C)$, $a \geq 1$ *and* $\deg B = b$.

i) If $a = 1$ D *is globally generated if and only if* $b + \mu^-(E) > 1$.
ii) If $a \geq 2$ D *is globally generated if* $b + a\mu^-(E) > 1$.

PROOF. To prove ii) it is sufficient to apply Proposition 2.2 and Lemma 2.4. To prove i) notice that if E is indecomposable it is enough to apply Proposition 2.2 and Lemma 2.3, observing that an indecomposable vector bundle E over an elliptic curve is semistable and hence $\mu^-(E) = \mu(E)$. Let now E be decomposable and hence $E \otimes B$ decomposable over C. In particular let $E \otimes B = \oplus_{q=1}^s A_q$ be a decomposition of $E \otimes B$ in indecomposable vector bundles A_q over C. By Lemma 2.3 every A_q, for $q = 1, \cdots, s$, is globally generated if and only if $\deg A_q > \operatorname{rk} A_q$, *i.e.* if and only if $\mu(A_q) > 1$, for all q. From Lemma 2.3, Lemma2.8 and Proposition 2.2 we get the following chain of equivalences which conclude the proof: $\mu^-(E) + b > 1 \Leftrightarrow \mu^-(E \otimes B) = \min_q \mu(A_q) > 1 \Leftrightarrow \mu(A_q) > 1$ for all $q \Leftrightarrow A_q$ is globally generated for all $q \Leftrightarrow \pi_* D$ is globally generated on $C \Leftrightarrow D$ is globally generated on X. ∎

The above lemma is partially contained in [6, Proposition 3.3]. Unfortunately the proof presented there is based on [6, Proposition 1.1(iv)], which is not correct, as the following counterexample shows. Let E be an indecomposable vector bundle over an elliptic curve with $\deg E = 1$ and $\operatorname{rank} E = 2$. Then $2\mathcal{T} = \mathcal{O}_{\mathbb{P}(E)}(2)$ is generated by global sections, according to [2, Proposition 8.5.8]. On the other hand let $\pi_*(2\mathcal{T}) = S^2E = \oplus_q A_q$, where A_q is indecomposable for all q. Then S^2E is generated by global sections if and only if A_q is such, for all q. From Lemma 2.3 it follows that S^2E is globally generated if and only if $\mu(A_q) > 1$ for all q, *i.e.* if and only if $\mu^-(S^2E) > 1$, *i.e.* if and only if $2\mu^-(E) = 2\mu(E) > 1$ which is false.

If we consider an indecomposable vector bundle of degree $d = 0$, we have the following proposition. It is contained in [7, Theorem 3.9], but we prefer to give here a simpler proof.

PROPOSITION 2.10. *Let E be an indecomposable* rank *r vector bundle over an elliptic curve C with* $\deg E = 0 \pmod r$*, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then D is very ample on X if and only if $b + a\mu^-(E) = b + a\mu(E) \geq 3$.*

PROOF. It is enough to consider the case in which E is normalized, as if E is not normalized we can consider its normalization $\bar{E} = E \otimes L$ with $\deg L = l$. If $D \equiv aT + bf$ in $\operatorname{Num} \mathbb{P}(E)$ then in $\operatorname{Num} \mathbb{P}(\bar{E})$ we get

$$D \equiv a\bar{T} + (b - al)\bar{f}, \quad \bar{d} = \deg \bar{E} = d + rl, \quad \mu(\bar{E}) = \mu(E) + al. \tag{3}$$

Let E be normalized, hence $d = 0$ and $E = F_r$ in the notation of [1] (recall that $F_1 = \mathcal{O}_C$). According to (2) the only cases to be considered are $b = 1$ and $b = 2$ and hence $D \equiv aT + f$ or $D \equiv aT + 2f$. We want to show that in both these cases D is not very ample. Assume the contrary and proceed by induction on r. Let $r = 2$. As $DT = 1$ or 2, the smooth elliptic curve Γ, which is the only element of $|\mathcal{T}|$, is embedded by $\phi_{|D|}$ as a line or a conic which is a contradiction. Assume now the proposition true for F_{r-1} and recall that there is a short exact sequence (see [1] p. 432)

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow 0. \tag{4}$$

Let $T' = T|_Y$ and $f' = f|_Y$ the generators of $\operatorname{Num}(Y)$ where $Y = \mathbb{P}(F_{r-1}) \subset X = \mathbb{P}(E)$. If D is very ample, $D|_Y$ is very ample too; but $D|_Y \equiv aT' + bf'$ and it is not very ample by induction hypothesis. Hence D is very ample if and only if $b \geq 3$. ∎

The following lemma, which gives a sufficient condition for the very ampleness of a divisor D on $X = \mathbb{P}(E)$, will be needed later on.

LEMMA 2.11. *Let E be a* rank r *vector bundle over a curve C and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$, with $a \geq 1$. If $\pi_* D$ is a very ample vector bundle on the curve C, then D is very ample on $\mathbb{P}(E)$. Moreover if $a = 1$, D is very ample on X if and only if $\pi_* D$ is very ample on C.*

PROOF. We give only a sketch of the proof. A divisor $D \equiv aT + bf$ on X defines a map $\varphi_{|D|}$ in a suitable projective space such that $X' = \varphi_{|D|}(X)$ is a bundle on C whose fibers are the Veronese embedding of the fibers of $X = \mathbb{P}(E)$. Moreover each fiber of X' is embedded in a fiber of the projective bundle $\mathbb{P}\big(S^a(E) \otimes \mathcal{O}_C(B)\big)$. It follows that the very ampleness of $S^a(E) \otimes \mathcal{O}_C(B)$ and hence of its tautological bundle implies that the map $\varphi_{|D|}$ gives an embedding and hence that D is very ample. The case $a = 1$ follows immediately from the above considerations. ∎

2.4. *The case $a = 1$.* We want to investigate the very ampleness of $D \equiv aT + bf$ in dependence of a and b. As we have remarked at the end of Section 2.2, the problem is open only when $0 < b + a\mu^-(E) \leq 2$. Let us begin with the case $a = 1$. In this case we have the following theorem:

THEOREM 2.12 (GUSHEL [6], THEOREM 4.3). *Let $D \sim \mathcal{T} + \pi^* B$ be a divisor on $\mathbb{P}(E)$, where E is an indecomposable and normalized vector bundle of* rank r *over an elliptic curve C. If $b = \deg B$, the divisor D is very ample if and only if:*

i) $b \geq 3$ *if* $\deg E = 0$,

ii) $b \geq 2$ *if* $0 < \deg E < r$.

Now it is easy to prove the following (see (3)):

PROPOSITION 2.13. *In the above assumptions and notations, if E is indecomposable but not normalized, it follows that D is very ample if and only if the following conditions hold:*

i) $b + \mu(E) \geq 3$ *if* $d = 0 \pmod r$.

ii) $b + \mu(E) \geq 2$ *otherwise.*

REMARK 2.14. The previous results consider the case in which E is indecomposable. If E is decomposable, by Lemma 2.12, we can argue as follows: firstly in this case, as $a = 1$, D is very ample if and only if $\pi_*(D)$ is very ample. Secondly we have $D \sim \mathcal{T} + \pi^* B$, $\pi_*(D) \simeq E \otimes \mathcal{O}_C(B) = \bigoplus E_j \otimes \mathcal{O}_C(B)$, with E_j indecomposable vector bundles. Moreover $E \otimes \mathcal{O}_C(B)$ is very ample if and only if every $E_j \otimes \mathcal{O}_C(B)$ is very ample. Let $\deg E_j = d_j$ and $\operatorname{rk} E_j = r_j$, and assume that $d_j = 0 \pmod{r_j}$, possibly only for $j = \cdots t$. Then D is very ample if and only if $b + \frac{d_j}{r_j} \geq 3$ for $j = 1 \cdots t$, and $b + \frac{d_j}{r_j} \geq 2$ for the remaining j's, by Proposition 2.13.

Having dealt above with the case $a = 1$, from now on the blanket assumption $a \geq 2$ will be in effect.

3. **Rank** 2. Let E be a rank 2 vector bundle on an elliptic curve C and let $D \equiv aT+bf$ be a line bundle on $X = \mathbb{P}(E)$. Assume that E is indecomposable. If E is normalized then $\deg E = 0, 1$ by Lemma 2.7. If $\deg E = 0$, from Proposition 2.10 it follows that D is very ample if and only if $b \geq 3$. If $\deg E = 1$, necessary and sufficient conditions for the very ampleness of D are given by the following theorem, reformulated under our assumption that $a \geq 2$.

THEOREM 3.1 (BIANCOFIORE-LIVORNI [3], THEOREM 6.3). *Let $D \sim a\mathcal{T} + \pi^*B$ be a divisor on $\mathbb{P}(E)$, where E is an indecomposable normalized vector bundle of rank* 2 *and degree* 1 *over an elliptic curve C. If $b = \deg B$, the divisor D is very ample if and only if $b + \frac{a}{2} > 1$.*

The following proposition can now be easily proved (see (3)).

PROPOSITION 3.2. *In the above hypothesis, if E is indecomposable but not normalized, D is very ample if and only if the following conditions hold:*

ii) $b + a\mu^-(E) \geq 3$ *if* $d = 0 \pmod 2$.

i) $b + a\mu^-(E) > 1$ *if* $d = 1 \pmod 2$.

The case E decomposable is treated by the following theorem.

THEOREM 3.3. *Let $D \sim a\mathcal{T} + \pi^*B$ be a divisor on $\mathbb{P}(E)$, where E is a decomposable vector bundle of* rank 2 *over an elliptic curve C, $b = \deg B$. The divisor D is very ample if and only if $b + a\mu^-(E) \geq 3$.*

PROOF. To prove the sufficient condition let E be decomposable as $H \oplus G$ where H and G are line bundles on C with $\deg H = h \geq \deg G = g$. By Lemma 2.8 it is $\mu^-(E) = g$. By Lemma 2.11, a sufficient condition for the very ampleness of D on X is that $\pi_*(D)$ is very ample as a vector bundle on C. In our hypothesis

$$\pi_*(D) = S^a(E) \otimes \mathcal{O}_C(B) = \bigoplus_{q=0}^{a} H^{\otimes q} \otimes G^{\otimes a-q} \otimes B.$$

Now $\pi_*(D)$ is very ample if each element of its decomposition has degree ≥ 3, *i.e.* if $qh+(a-q)g+b \geq 3$, for all $q = 0, \ldots, a$. As the minimum of $qh+(a-q)g+b$ is realized for $q = 0$, $\pi_*(D)$ is very ample if and only if $ag + b = b + a\mu^-(E) \geq 3$. This condition is also necessary for the very ampleness of D. Indeed the projective bundle $\mathbb{P}(G)$, by the exact sequence

$$0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0$$

[6], Proposition 1.1, gives an elliptic curve Γ on X, $\Gamma \in |\mathcal{T} + \pi^*(H^*)|$. Notice that $h^0\big(X, \mathcal{T} + \pi^*(H^*)\big) > 0$. If D is very ample it must be $D\Gamma = (aT+bf)(T-hf) = ag+b = b + a\mu^-(E) \geq 3$. ■

4. **Rank** 3. In this section and in the next one, we will prove the very ampleness of a divisor on a smooth variety following a classical method, based on the following lemmata.

LEMMA 4.1. *Let X be a smooth variety, D a divisor in* $\mathrm{Pic}(X)$ *and let A be an other element of* $\mathrm{Pic}(X)$, *such that $h^1\big(X, \mathcal{O}_X(D-A)\big) = 0$. If, for each pair of points $R, Q \in X$ (possibly infinitely near) it is possible to find a smooth element $S \in |A|$ containing R and Q, then D is very ample on X if and only if $D|_S$ is very ample on S.*

PROOF. If D is very ample then obviously $D|_S$ is very ample. On the other hand, pick any two points $R, Q \in X$ (distinct or infinitely near) and choose $S \in |A|$ such that $R, Q \in S$. As $D|_S$ is very ample there exist sections of $D|_S$ separating R and Q. Now look at the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(D-A) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_S(D|_S) \longrightarrow 0.$$

From the assumptions above we get that the map $H^0\big(X, \mathcal{O}_X(D)\big) \longrightarrow H^0\big(S, \mathcal{O}_S(D|_S)\big)$ is surjective and hence D is very ample on X if and only if $D|_S$ is very ample on S. ■

LEMMA 4.2. *Let E be an ample vector bundle over an elliptic curve C such that $\deg E < \operatorname{rk} E$. Let $X = \mathbb{P}(E)$, let P be a fixed point of C, $D \sim a\mathcal{T} + \pi^*B$ and $A = \mathcal{B} + \mathcal{F}_P$ be line bundles on X, with $b = \deg B$, $b-1+(a-1)\mu^-(E) > 0$ and $h^0\big(X, \mathcal{O}_X(A)\big) \geq \deg E+3$. Then the hypothesis of Lemma 4.1 are satisfied for A.*

PROOF. If $(a-1)\mu^-(E)+b-1 > 0$ then by Lemma 2.4 it is $h^1\big(X, \mathcal{O}_X(D-A)\big) = 0$. Moreover being $h^0\big(X, \mathcal{O}_X(A)\big) \geq \deg E + 3$, for each pair of points $R, Q \in X$ there exists a linear subsystem $\mathcal{L} \subset |\mathcal{A}|$, with $\dim \mathcal{L} \geq \deg \mathcal{E}$, all the elements of which contain R and Q. Moreover in $\mathcal{L}$ there is at least one smooth element S. In fact, assume that all the elements of $\mathcal{L}$ are singular. Note that any singular element of $\mathcal{L}$ must be reducible as $\Gamma \cup \mathcal{F}_P$, with $\Gamma \in |\mathcal{T}|$, Γ smooth, because we have that any divisor numerically equivalent to $T-f$ is not effective as $\deg E < \operatorname{rk} E$. As $h^0\big(X, \mathcal{O}_X(\mathcal{F}_P)\big) = 1$, for all $P \in C$, by Bertini's theorem all the elements of $\mathcal{L}$ are singular only if $\mathcal{F}_P$ is fixed and Γ varies in a subsystem of $|\mathcal{T}|$ of dimension $\deg E$. This is impossible as $h^0\big(X, \mathcal{O}_X(\mathcal{T})\big) = \deg E$. ■

LEMMA 4.3. *Let E be an ample vector bundle over an elliptic curve C such that $\deg E < \operatorname{rk} E$. Let $X = \mathbb{P}(E)$, and let $D \sim a\mathcal{T} + \pi^*B$ be a line bundle on X, with $b = \deg B$, $b + (a-1)\mu^-(E) > 0$ and $h^0\big(X, \mathcal{O}_X(\mathcal{T})\big) \geq 3$. Then the hypothesis of Lemma 4.1 are satisfied, with $A = \mathcal{T}$.*

PROOF. If $(a-1)\mu^-(E) + b > 0$ then Lemma 2.4 gives $h^1\big(X, \mathcal{O}_X(D-A)\big) = 0$. Moreover as each element of $|\mathcal{T}|$ is smooth, because we have that any divisor numerically equivalent to $T-f$ is not effective as $\deg E < \operatorname{rk} E$, the condition $h^0\big(X, \mathcal{O}_X(\mathcal{T})\big) \geq 3$ shows that it is possible to find a smooth element $S \in |\mathcal{T}|$ containing each fixed pair of points $R, Q \in X$. ■

The following lemma will be very useful to obtain the vanishing condition required by Lemma 4.1 in many borderline cases. The notation used here is the classical notation used by Atiyah in [1].

LEMMA 4.4. *Let E be an indecomposable vector bundle over an elliptic curve C with* $\operatorname{rank} E = r$ *and* $\deg E = d$. *Let* $X = \mathbb{P}(E)$ *and let* $\pi : X \longrightarrow C$ *be the natural projection. Let* $D = a\mathcal{T} + \pi^*(B)$ *for a line bundle B with* $\deg B = b$ *and let* $A = \mathcal{T} + \pi^*\big(\mathcal{O}_C(P)\big)$ *where P is a point in C.*

If $\frac{(a-1)d}{r} + b - 1 = 0$, *it is possible to choose* $P \in C$ *such that* $h^1(X, D - A) = 0$.

PROOF. It is enough to show that $h^1\big(C, S^{a-1}E \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$.

Since $\deg S^{a-1}E \otimes B \otimes \mathcal{O}_C(-P) = 0$ by Riemann Roch it is enough to show that $h^0\big(S^{a-1}E \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$. Because $S^{a-1}(E)$ is a direct summand of $E^{\otimes(a-1)}$ it is enough to show that $h^0\big(E^{\otimes(a-1)} \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$.

Let $h = \gcd(d, r)$. Then by [1] Lemma 2.4 and 2.6 it is

$$E = E' \otimes F_h \tag{5}$$

where $d' = \deg E' = \frac{d}{h}$ and $r' = \operatorname{rank} E' = \frac{r}{h}$ so that $\gcd(d', r') = 1$, F_h is as in [1] Theorem 5 and E' is indecomposable. Being r' and d' relatively prime, the condition $\frac{(a-1)d'}{r'} + b - 1 = 0$ shows that r' divides $(a-1)$. Therefore following [8] Proposition 1.4 it follows that

$$E'^{\otimes(a-1)} = \bigoplus_i (F_{r_i} \otimes L_i) \tag{6}$$

Therefore putting (5) and (6) together we get

$$E^{\otimes(a-1)} = (E' \otimes F_h)^{\otimes(a-1)} = \bigoplus_i (F_{r_i} \otimes L_i) \otimes F_h^{\otimes(a-1)}.$$

Theorem 8 in [1] shows that tensor powers of F_l 's are direct sums of F_k's so we conclude that

$$E^{(a-1)} = \bigoplus_j (F_{r_j} \otimes L_j).$$

It is then enough to show that for all j it is $h^0\big(F_{r_j} \otimes L_j \otimes B \otimes \mathcal{O}_C(-P)\big) = 0$.

Let $\mathcal{L}_{j,P} = L_j \otimes B \otimes \mathcal{O}_C(-P)$. Recall that the F_{r_j} are obtained as successive extensions of each other by $\mathcal{O}_C$, *i.e.* for every r we have the sequence (4) (see proof of Proposition 2.11).

This shows that it is $h^0(F_{r_j} \otimes \mathcal{L}_{j,P}) = 0$ unless $\mathcal{L}_{j,P} = \mathcal{O}_C$. It is then enough to choose a point P such that $L_j \otimes B \otimes \mathcal{O}_C(-P) \neq \mathcal{O}_C$ for all j. Since B is a fixed line bundle and j runs over a finite set, a P that works for all j can certainly be found. ∎

The following theorem collects our results for the case $\operatorname{rk} E = 3$.

THEOREM 4.5. *Let E be a* rank 3 *vector bundle on an elliptic curve C and let* $D \equiv aT + bf$ *be a line bundle on* $X = \mathbb{P}(E)$.

(a) *If E is indecomposable then*
  (a1) *if* $d = 0 \pmod 3$, *D is very ample if and only if* $b + a\mu^-(E) \geq 3$,
  (a2) *if* $d = 1 \pmod 3$, *D is very ample if* $b + a\mu^-(E) > 1$,
  (a3) *if* $d = 2 \pmod 3$, *D is very ample if* $b + a\mu^-(E) > \frac{4}{3}$,

(b) *if E is decomposable, then D is very ample if and only if* $b + a\mu^-(E) \geq 3$ *except when* $E = E_1 \oplus E_2$, *with* $\operatorname{rk} E_1 = 1$, $\operatorname{rk} E_2 = 2$, $\deg E_2$ *odd and* $\deg E_1 > \frac{\deg E_2}{2}$. *In the latter case the condition is only sufficient.*

PROOF. Firstly we consider the case E indecomposable and normalized. By Lemma 2.7 and Proposition 2.10 only the cases $d = 1$ and $d = 2$ need to be considered.

CASE 1. $d = 1$.

Let A be as in Lemma 4.2 and notice that $h^0(A) = \deg E + 3$. By Remark 2.6. We can assume $b+\frac{a}{3} = 1+\frac{\varepsilon}{3}$, where $\varepsilon = 1, 2, 3$. If $\varepsilon = 2, 3$ then $(a-1)\mu^-(E)+b-1 = b+\frac{a}{3}-\frac{4}{3} > 0$. If $\varepsilon = 1$ Lemma 4.4 still allows the use of Lemma 4.1. Therefore Lemma 4.1 and 4.2 can be applied. Let S be a smooth element in $|\mathcal{T} + \mathcal{F}_P|$. It is enough to check when $D|_S$ is very ample. By [6], Proposition 1.1, $S = \mathbb{P}(E')$ where E' is a rank 2 vector bundle on the curve C, with $\deg E' = 2$, defined by the sequence

$$(7) \qquad 0 \longrightarrow \mathcal{O}_C(-P) \longrightarrow E \longrightarrow E' \longrightarrow 0.$$

As Num(S) is generated by T' and f', with $T' = T|_S$ and $f' = f|_S$, $D|_S \equiv aT' + bf'$ and moreover by Section 3, $D|_S$ is very ample in our range if and only if $b + a\mu^-(E') \geq 3$. If E' is indecomposable then $\mu^-(E') = \mu(E') = 1$. If E' is decomposable, *i.e.* $E' = H \oplus G$, both $\mu(H) \geq \mu(E) = \frac{1}{3}$ and $\mu(G) \geq \mu(E) = \frac{1}{3}$ because from (7) there exist non zero surjective morphisms $E \longrightarrow H$ and $E \longrightarrow G$.

Hence $\deg H = \deg G = 1$, $\mu(H) = \mu(G) = 1$ and $\mu^-(E') = 1$. In every case the condition $b + a\mu^-(E') \geq 3$ is satisfied in the range under consideration.

CASE 2. $d = 2$.

Let A and S be as in Lemma 4.1 and 4.2. In this case it is $h^0(X, A) = 5 \geq \deg E + 3$. By Remark 2.6 we can assume $b + a\frac{2}{3} = 1 + \frac{\varepsilon}{3}$ with $\varepsilon = 1, 2, 3$. If $\varepsilon = 3$ the hypothesis of Lemma 4.2 are satisfied. If $\varepsilon = 2$ Lemma 4.4 still allows the use of Lemma 4.1. Therefore it suffices to investigate the very ampleness of $D|_S$. Let $S = \mathbb{P}(E')$ with E' a rank 2 vector bundle on C, with $\deg E' = 3$ defined again by (7). If E' is indecomposable, $D|_S \equiv aT' + bf'$ is very ample if and only if $b + a\mu^-(E') > 1$, *i.e.* $b + \frac{3}{2}a > 1$ *i.e.* for all D in the range under consideration.

If E' is decomposable then $E' = H \oplus G$, with $\deg H$ and $\deg G \geq \mu(E) = \frac{2}{3}$. Hence we can assume $\deg H = 1$ and $\deg G = 2$. By Lemma 2.8, $\mu^-(E) = 1$ and the very ampleness condition is $b + a \geq 3$ which is satisfied by every D in the range under consideration.

Notice that in the case $b + \frac{2}{3}a = \frac{4}{3}$, *i.e.* $\varepsilon = 1$, a very ampleness result for all D in our range cannot be expected. For example, $D \sim 2\mathcal{T}$ is not very ample as $D|_Y$ is not very ample for each smooth surface $Y \in |\mathcal{T}|$ by Section 3.

If E is indecomposable but not normalized, the result is obtained by similar computations (see (3)).

To prove (b), let now E be decomposable. Firstly we prove the sufficient condition. By Proposition 2.2 it suffices to prove that $\pi_*(D)$ is very ample. Let us consider $\pi_*(D) = S^a(E) \otimes \mathcal{O}_C(B) = \oplus_q A_q$ where A_q is an indecomposable vector bundle on C for all q. $S^a(E) \otimes \mathcal{O}_C(B)$ is very ample if and only if A_q is very ample for all q *i.e.* if $\mu(A_q) \geq 3$ for all q. This condition is satisfied if $\min_q \mu(A_q) = \mu^-\big(S^a(E) \otimes O_C(B)\big) = b + a\mu^-(E) \geq 3$ which is what we wanted to show.

To prove the necessary condition, two cases will be considered:

i) E is sum of three line bundles, $E = W \oplus G \oplus H$ respectively of degrees $w \leq g \leq h$. By Lemma 2.8 $\mu^-(E) = w$, and $d = \deg E = w + g + h$. From [6], Proposition 1.1 it follows that $\mathbb{P}(G \oplus H)$ is a subvariety of X corresponding to a line bundle numerically equivalent to $T - wf$ while $\mathbb{P}(W)$ is an elliptic curve Γ on X, isomorphic to C, which is numerically equivalent to $T^2 + xTf$, for some $x \in \mathbb{Z}$. As the cycles corresponding to $\mathbb{P}(W)$ and $\mathbb{P}(G \oplus H)$ do not intersect, from $(T^2 + xTf)(T - wf) = 0$ we get $x = -(g + h)$. If D is a very ample line bundle on X, $D|_\Gamma$ is very ample, hence $D\Gamma = b + a\mu^-(E) \geq 3$.

ii) $E = H \oplus G$ where $\operatorname{rk} H = 1$, $\operatorname{rk} G = 2$, $h = \deg H$, $g = \deg G$. As in i), we get that $Z = \mathbb{P}(H)$ is numerically equivalent to $T^2 - gTf$ and the very ampleness of $D|_Z$ implies $b + ah \geq 3$. If $h \leq \frac{g}{2}$ this concludes the proof. Otherwise let us denote by Y the smooth surface $\mathbb{P}(G)$. As usual $\mathrm{Num}(Y)$ is generated by $T' = T|_Y$ and $f' = f|_Y$. The very ampleness of D implies the one of $D|_Y$ and by Section 3, $D|_Y$ is very ample if and only if

$b + a\frac{g}{2} \geq 3$ if g is even and,

$b + a\frac{g}{2} > 1$ if g is odd.

If g is even, a necessary condition for the very ampleness of D is $b + a\frac{g}{2} \geq 3$ *i.e.* $b + a\mu^-(E) \geq 3$ which is the desired condition.

If g is odd, necessary conditions for the very ampleness of D are both $b + ah \geq 3$ and $b + a\frac{g}{2} > 1$. Hence only the sufficient condition $b + a\mu^-(E) \geq 3$ is obtained in this case. ■

5. **Rank** r.

To deal with the case of E vector bundle on an elliptic curve C, of rank $r > 3$ we need to recall first the following

THEOREM 5.1 ([8]). *Let E be a vector bundle of* rank r *on an elliptic curve C. E is ample if and only if every indecomposable direct summand E_i of E has* $\deg E_i > 0$.

Our method based on Lemma 4.1, 4.2 and 4.3 forces us to investigate first the case $\deg E = 3$, then $\deg E = 1, 2$ and finally $\deg E \geq 4$.

5.1. $\deg E = 3$.

THEOREM 5.2. *Let E be a vector bundle over an elliptic curve C, with* $\deg E = 3$ *and* rank $E = 4$, *and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

i) If E is indecomposable, and $b + \frac{3}{4}a > \frac{3}{4}$, D is very ample if and only if $b + a \geq 3$.

ii) If E is decomposable and ample, and $b + \frac{a}{3} > \frac{1}{3}$, D is very ample if $b + \frac{a}{2} > 2$.

PROOF. i) If $\deg E = h^0(\mathcal{T}) = 3$ then we can apply Lemma 4.3 and 4.1 with $A = \mathcal{T}$, as $b + (a-1)\mu^-(E) = b + \frac{3}{4}a - \frac{3}{4} > 0$ by hypothesis. Hence D is very ample if and only if $D|_S$ is very ample, where S is a suitable element of $|\mathcal{T}|$. There exists a vector bundle E' with $\operatorname{rk} E' = 3$, $\deg E' = 3$, given by

$$0 \longrightarrow O_C \longrightarrow E \longrightarrow E' \longrightarrow 0, \tag{8}$$

such that $S = \mathbb{P}(E')$ and $\mathrm{Num}(S)$ is generated by T' and f', where $T' = T|_S$ and $f' = f|_S$ so that $D|_S \equiv aT' + bf'$.

By Section 4, $D|_S$ is very ample if and only if $b+a \geq 3$. Indeed if E' is indecomposable the necessary and sufficient condition for the very ampleness is $b + a \geq 3$. If $E' = \oplus_i E'_i$ then $\mu(E'_i) \geq \frac{3}{4}$ for all i. Hence the only possibilities for a decomposition of E' are:

1) $E' = F \oplus G \oplus H$, with F, G, H line bundles all of degree 1.

2) $E' = H \oplus G$, with $\operatorname{rank} G = 2$, $\operatorname{rank} H = 1$, $\deg G = 2$, $\deg H = 1$.

Again by Section 4 in both the above cases, the necessary and sufficient condition for the very ampleness of $D|_S$ is $b + a \geq 3$.

ii) Let us suppose that E is decomposable and ample. Then the possible decompositions for E give $\mu^-(E) = \frac{2}{3}, \frac{1}{3}, \frac{1}{2}$. Note that the condition $b + \frac{a}{3} > \frac{1}{3}$ allows us to apply Lemma 4.3 with $A = \mathcal{T}$ in any case. If S is the usual element of $|\mathcal{T}|$, we get that D is very ample on X if and only if $D|_S$ is. If $S = \mathbb{P}(E')$, E' could be decomposable. In this case $\mu^-(E') \geq \frac{1}{2}$. Therefore the condition $b + \frac{a}{2} > 2$ guarantees the very ampleness of $D|_S$ by Section 4. ∎

Because $a \geq 2$, Theorem 5.2 immediately gives the following:

COROLLARY 5.3. *Let E be an ample vector bundle over an elliptic curve C, with $\deg E = 3$ and $\operatorname{rank} E = 4$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$ with $b + \frac{a}{3} > \frac{1}{3}$. If $b + \frac{a}{2} > 2$ then D is very ample.*

THEOREM 5.4. *Let E be an ample vector bundle over an elliptic curve C, with $\deg E = 3$ and $r = \operatorname{rank} E \geq 4$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$ with $b + a\mu^-(E) > \frac{3}{5}$. Then D is very ample if $b + \frac{a}{2} > 2$ and $b + \frac{a}{3} > \frac{1}{3}$.*

PROOF. Proceed by induction on $r = \operatorname{rank} E$. If $r = 4$ the inductive hypothesis is verified by Corollary 5.3. Let $r \geq 5$. It is $h^0(X, \mathcal{T}) = h^0(C, E) = 3$. Notice that $\mu^-(E) \leq \mu(E) \leq \frac{3}{r} \leq \frac{3}{5}$. Therefore $b + (a-1)\mu^-(E) > 0$ and Lemma 4.3 can be applied, with $A = \mathcal{T}$. Let $S = \mathbb{P}(E')$ be as in (8) with $\deg(E') = 3$ and $\mathrm{rk}(E') = r - 1 \geq 4$. $\mathrm{Num}(S)$ is generated by $T' = T|_S$ and $f' = f|_S$, so that $D|_S \equiv aT' + bf'$. Notice that E' is ample being a quotient of E. Notice that $\mu^-(E) \leq \mu^-(E')$. Indeed there exists a map from at least one direct summand E_k of E and the summand E'_j of E' which realizes $\mu^-(E')$ and so we get $\mu^-(E) \leq \mu(E_k) \leq \mu(E_j) = \mu^-(E')$. As $b + a\mu^-(E') \geq b + a\mu^-(E) > \frac{3}{5}$, by induction $D|_S$ is very ample if $b + \frac{a}{2} > 2$, $b + \frac{a}{3} > \frac{1}{3}$. ∎

5.2. $\deg E = 2$.

THEOREM 5.5. *Let E be an indecomposable vector bundle over an elliptic curve C, with $\deg E = 2$, $\mathrm{rk}\, E = r \geq 4$. Let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

i) If $r = 4$, D is very ample if $b + \frac{a}{2} > 2$.

ii) If $r \geq 5$, D is very ample if $b + \frac{a}{2} > 2$, $b + \frac{a}{3} > \frac{1}{3}$ and $b + a\frac{2}{r} > 1 + \frac{1}{r}$.

PROOF. Let A be as in Lemma 4.2 and notice that $h^0(A) = 2 + r > \deg E + 3$. By Remark 2.6 and the assumptions in i) and ii) we can assume

$$b + a\frac{2}{r} = 1 + \frac{\varepsilon}{r}, \tag{9}$$

where $\varepsilon = 1, 2 \ldots, r$. If $b + a\frac{2}{r} > 1 + \frac{2}{r}$ the assumptions of Lemma 4.1 and 4.2 are satisfied. If $b + a\frac{2}{r} = 1 + \frac{2}{r}$ the line bundle $D - A$ has degree 0 and Lemma 4.4 shows that Lemma 4.1 can be used if

$$b + a\frac{2}{r} \geq 1 + \frac{2}{r}. \tag{10}$$

Therefore condition (10) can be rewritten using (9) as

$$b + a\frac{2}{r} > 1 + \frac{1}{r}. \tag{11}$$

Let $S = \mathbb{P}(E')$, where E' is as in (7), where $\deg E' = 3$, $\operatorname{rk} E' = r - 1 \geq 3$, Num($S$) is generated by $T' = T|_S$ and $f' = f|_S$, and $D|_S \equiv aT' + bf'$. Notice that E' is indecomposable or decomposable and ample because $\mu^-(E') \geq \mu^-(E) = \frac{2}{r} > 0$.

To prove i) assume $r = 4$. In this case $\operatorname{rk} E' = 3$ and $\deg E' = 3$. If E' is indecomposable, as $b + a\mu^-(E') > b + a\mu^-(E) = b + \frac{a}{2} > 1$, by Theorem 4.5 $D|_S$ is very ample if $b + a \geq 3$. By the same theorem if E' is decomposable and ample then $D|_S$ is very ample if $b + a\mu^-(E') > 2$. The possible values for $\mu^-(E')$ are 1 or $\frac{1}{2}$. Therefore by Theorem 4.5 it follows that $D|_S$ is very ample if

$$b + \frac{a}{2} > 2. \tag{12}$$

Putting (11) and (12) together it follows that in the case $r = 4$ D is very ample if $b + \frac{a}{2} > \frac{5}{4}$ and $b + \frac{a}{2} > 2$ *i.e.* $b + \frac{a}{2} > 2$.

To prove ii) assume $r - 1 \geq 4$. Since $b + a\mu^-(E') \geq b + a\mu^-(E) = b + a\frac{2}{r} > 1 + \frac{1}{r} > \frac{3}{5}$ from Theorem 5.4 it follows that $D|_S$ is very ample if $b + \frac{a}{2} > 2$ and $b + \frac{a}{3} > \frac{1}{3}$. Therefore if $r \geq 5$, D is very ample if $b + \frac{a}{2} > 2$, $b + \frac{a}{3} > \frac{1}{3}$ and $b + a\frac{2}{r} > 1 + \frac{1}{r}$. ■

THEOREM 5.6. *Let E be a decomposable and ample vector bundle of* rank $r \geq 4$ *over an elliptic curve C, with* $\deg E = 2$. *Let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

i) If $r = 4$, D is very ample if $b + \frac{a}{2} > 2$ and $b + a\mu^-(E) > \frac{3}{2}$.

ii) If $r \geq 5$, D is very ample if $b + a\mu^-(E) > 1 + \frac{2}{r}$, $b + \frac{a}{3} > \frac{1}{3}$ and $b + \frac{a}{2} > 2$.

PROOF. Let A be as in Lemma 4.2 and notice that $h^0(A) = r + 2 \geq \deg E + 3$. Also notice that $\mu^-(E) \leq \mu(E) = \frac{2}{r}$. In our hypothesis we have $b + (a - 1)\mu^-(E) - 1 \geq b + a\mu^-(E) - \frac{2}{r} - 1 > 0$.

Therefore Lemma 4.1 and 4.2 can be applied. Let $S = \mathbb{P}(E')$, where E' is as in (7) where $\deg E' = 3$ and $\operatorname{rk}(E') = r - 1 \geq 3$. If $r \geq 5$ we can apply Theorem 5.4 to $D|_S$. Indeed $b + a\mu^-(E') \geq b + a\mu^-(E) > 1 + \frac{2}{r} > \frac{3}{5}$. Hence $D|_S$ is very ample if $b + \frac{a}{3} > \frac{1}{3}$ and $b + \frac{a}{2} > 2$.

If $r = 4$ we can apply Theorem 4.5. If E' is indecomposable then $D|_S$ is very ample if $b+a \geq 3$. If E' is decomposable, noticing that E' is ample, the condition is $b+a\mu^-(E') > 2$. In this case $\mu^-(E')$ can be 1 or $\frac{1}{2}$ and hence the worst sufficient condition becomes $b+\frac{a}{2} > 2$. ■

Theorem 5.5 and 5.6 give the following corollary.

COROLLARY 5.7. *Let E be an ample vector bundle over an elliptic curve C, with* $\deg E = 2$ *and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. Then the following conditions hold:*

i) If $\operatorname{rank} E \geq 5$, *then D is very ample if* $b+a\mu^-(E) > 1+\frac{2}{r}$, $b+\frac{a}{2} > 2$ *and* $b+\frac{a}{3} > \frac{1}{3}$.

ii) If $\operatorname{rank} E = 4$, *$D$ is very ample if* $b + a\mu^-(E) > \frac{3}{2}$ *and* $b+\frac{a}{2} > 2$.

5.3. $\deg E = 1$. Note that if $\deg E = 1$, E is ample if and only if it is indecomposable.

THEOREM 5.8. *Let E be an indecomposable vector bundle over an elliptic curve C, with* $\deg E = 1$, $\operatorname{rk} E = r \geq 4$, *and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$ with* $b+\frac{a}{r} > 1$. *Then the following conditions hold:*

i) If $r = 4$, D is very ample if $b+\frac{a}{2} > 2$.

ii) If $r = 5$, D is very ample if $b+\frac{a}{3} > \frac{3}{2}$ *and* $b+\frac{a}{2} > 2$.

iii) If $r \geq 6$, D is very ample if $b+\frac{a}{r-2} > 1+\frac{2}{r-1}$ *and* $b+\frac{a}{2} > 2$.

PROOF. In our hypothesis it is $(a-1)\frac{1}{r} + b - 1 > 0$. Lemma 4.4 allows us to apply Lemma 4.1 and 4.2 when $(a-1)\frac{1}{r} + b - 1 \geq 0$, noticing that if $A = \mathcal{T} + \mathcal{F}_P$ it is $h^0(A) = 1 + r \geq 5$. If $S = \mathbb{P}(E')$, where E' is as in (7), notice that E' is ample because E is. It is $\operatorname{rk} E' = r - 1 \geq 3$ and $\deg E' = 2$. Let us now distinguish the three cases according to the values of r. Theorem 5.4 and Corollary 5.7 will be used.

i) If rank $E = 4$, rank $E' = 3$, and $D|_S$ is very ample if $b + \frac{2}{3}a > \frac{4}{3}$ when E' is indecomposable while if E' is decomposable and ample the condition is $b+a\mu^-(E') > 2$. In the worst case $\mu^-(E') = \frac{1}{2}$ so we have $b+\frac{a}{2} > 2$.

ii) If rank $E = 5$, rank $E' = 4$, and $D|_S$ is very ample if $b+a\mu^-(E') > \frac{3}{2}$, and $b+\frac{a}{2} > 2$. As E' is indecomposable or decomposable and ample, in the worst case $\mu^-(E') = \frac{1}{3}$ and so it is enough to ask $b+\frac{a}{3} > \frac{3}{2}$ and $b+\frac{a}{2} > 2$.

iii) If rank $E \geq 6$ then rank $E' \geq 5$, hence by Corollary 5.7 $D|_S$ is very ample if $b+a\mu^-(E') > 1+\frac{2}{r-1}$, $b+\frac{a}{3} > \frac{1}{3}$ and $b+\frac{a}{2} > 2$. If E' is indecomposable then $\mu^-(E') = \frac{2}{r-1}$ while if E' is decomposable and ample, then $\mu^-(E') = \min(\frac{1}{s}, \frac{1}{r-1-s})$ with $s = 1 \cdots r-2$. So the condition $b + a\mu^-(E') > 1 + \frac{2}{r-1}$ can be substituted by $b + \frac{a}{r-2} > 1 + \frac{2}{r-1}$ which implies the condition $b+\frac{a}{3} > \frac{1}{3}$. ■

5.4. $\deg E \geq 4$.

THEOREM 5.9. *Let E be an ample vector bundle over an elliptic curve C, with* $\deg E = d \geq 4$, $\operatorname{rk} E = r \geq 4$ *and $d < r$. Let $D \equiv aT + bf$ be a line bundle on*

$X = \mathbb{P}(E)$. If $b + \frac{a}{d-1} > 2$ and $b + (a-1)\mu^-(E) > 0$ then D is very ample.

PROOF. The proof proceeds by induction on r. If $r = 4$ the smallest possible value of $\mu^-(E)$ is $\frac{1}{3}$. Since $d \geq 4$ it is $\frac{1}{d-1} \leq \frac{1}{3}$. Therefore $b + a\mu^-(E) \geq b + \frac{a}{3} \geq b + \frac{a}{d-1} > 2$. Lemma 2.11 and Theorem 2.5 conclude the proof of the initial inductive step. Let us suppose the statement true for $r-1$ and prove it for r. Let A be as in Lemma 4.3 and notice that $h^0(A) = d \geq 4$. Therefore Lemma 4.3 and 4.1 can be applied. By considering $S = \mathbb{P}(E')$ where E' is as in (8), we get that E' is ample with $\deg E' = d \geq 4$, $\operatorname{rk} E' = r' = r - 1 \geq 4$. Because $\mu^-(E') \geq \mu^-(E)$ it is $b + (a-1)\mu^-(E') \geq b + (a-1)\mu^-(E) > 0$ and again $b + \frac{a}{d-1} > 2$. Hence we can conclude by induction hypothesis. ∎

REMARK 5.10. Note that in the above theorem, if E is indecomposable, by normalizing E we can always assume $d < r$.

In a very particular case we can say a little more:

PROPOSITION 5.11. *Let E be an indecomposable vector bundle over an elliptic curve C, with $\deg E = d \geq 4$, $\operatorname{rk} E = d + 1$, and let $D \equiv aT + bf$ be a line bundle on $X = \mathbb{P}(E)$. If $b + (a-1)\frac{d}{d+1} > 0$ and $b + a > 2$ then D is very ample.*

PROOF. It is $h^0(X, \mathcal{T}) = d \geq 4$. Our hypothesis $b + (a-1)\frac{d}{d+1} > 0$ shows that Lemma 4.1 and 4.3 can be applied.

By considering $S = \mathbb{P}(E')$, where E' is as in (8) it is $\deg E' = \operatorname{rk} E' = d$. A sufficient condition for the very ampleness of $D|_S$ is $b + a\mu^-(E') > 2$ by Lemma 2.11 and Theorem 2.5. We claim that in this case $\mu^-(E') = 1$. If E' is indecomposable it is $\mu^-(E') = \mu(E') = 1$. If E' is decomposable we can suppose that $E' = \oplus G_j$, with $r_j = \operatorname{rk} G_j \geq 1$, $d_j = \deg G_j \geq 1$ (as $\mu(G_j) \geq \mu(E) = \frac{d}{d+1}$) and $\sum d_j = \sum r_j = d$. Hence we have $\frac{d_j}{r_j} \geq \frac{d}{d+1}$, for all j, which implies $r_j - d_j \leq \frac{d_j}{d} < 1$, for all j. Hence $r_j - d_j \leq 0$, for all j *i.e.* $d_j = r_j + s_j$ with $s_j \geq 0$, for all j. But $d = \sum d_j = \sum(d_j + s_j) = d + \sum s_j$ and $\sum s_j = 0$ *i.e.* $s_j = 0$, for all j. Hence $\mu^-(E') = 1$. ∎

REFERENCES

1. M. F. Atiyah, *Vector bundles over an elliptic curve*, Proc. London Math. Soc. (3) **7**(1957), 414–452.
2. M. C. Beltrametti and A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, Exposition. Math. **16**, De Gruyter, 1995.
3. A. Biancofiore and E. L. Livorni, *On the Genus of a Hyperplane Section of a Geometrically Ruled Surface*, Ann. Mat. Pura Appl. **147**(1987), 173–185.
4. D. C. Butler, *Normal generation of vector bundles over a curve*, J. Differential Geom. (1) **39**(1994), 1–34.
5. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.
6. N. P. Gushel, *Very ample divisors on projective bundles over an elliptic curve*, Mat. Zametki (6) **47**(1990), 15–22.
7. ———, *Very ample divisors on projective bundles over curves*, St. Petersburg Math. J. (2) **4**(1993), 297–307.
8. R. Hartshorne, *Ample vector bundles on curves*, Nagoya Math. J. **43**(1971), 41–72.

9. ———, *Algebraic Geometry*, Graduate Texts in Math. **52**, Springer Verlag, New York, Heidelberg, Berlin, 1977.

10. Y. Miyaoka, *The Chern class and Kodaira dimension of a minimal variety*. In: Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. **10**, Amer. Math. Soc., 1987, 449–476.

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NEW ALMOST PERIODIC TYPE FUNCTIONS AND SOLUTIONS OF DIFFERENTIAL EQUATIONS

BOLIS BASIT AND CHUANYI ZHANG

ABSTRACT. Let X be a Banach space and $\mathbf{J} \in \{\mathbf{R}^+, \mathbf{R}\}$. Let Π and Π_0 be two subspaces of $C(\mathbf{J}, X)$, the Banach space of bounded continuous functions from $\mathbf{J}$ to X. We seek conditions under which $\Pi + \Pi_0$ is closed in $C(\mathbf{J}, \mathbf{R})$. This led to introduce a general $\mathcal{A}\Pi_r(\mathbf{J}, X)$ space, which contains many classes of almost periodic type functions as subspaces. We prove some recent results on indefinite integral for the elements of these classes. We apply certain results on harmonic analysis to investigate solutions of differential equations. As an application we study specific $\mathcal{A}\Pi_r(\mathbf{J}, \mathbf{R})$ spaces: the spaces of asymptotic and pseudo almost automorphic functions and their solutions of some ordinary quasi-linear and a non-linear parabolic partial differential equations.

1. **Notation, Definitions and Introduction.** Throughout this paper, X denotes a Banach space and X^* denotes the dual space of X. Let $\mathbf{J} \in \{\mathbf{R}^+, \mathbf{R}\}$. Then $C(\mathbf{J}, X)$ will stand for the Banach space of bounded continuous functions φ from $\mathbf{J}$ to X with norm $\|\varphi\| = \sup_{x\in \mathbf{J}} \|\varphi(x)\|$ and $C_u(\mathbf{J}, X)$ will denote the subspace of $C(\mathbf{J}, X)$ consisting of the uniformly continuous functions. If $(\varphi_n), \varphi \subset C(\mathbf{J}, X)$, we write $\varphi_n \longrightarrow \varphi$ if and only if $\|\varphi_n - \varphi\| \longrightarrow 0$ as $n \longrightarrow \infty$. In case that $X = \mathbf{C}$, we will omit X from our notation and write, say, $C(\mathbf{J})$ for $C(\mathbf{J}, X)$. The translate and the difference of φ by $s \in \mathbf{J}$ are respectively the functions $R_s\varphi(t) := \varphi(t+s)$ for all $t \in \mathbf{J}$ and $\Delta_s\varphi := R_s\varphi - \varphi$. Let $\mathcal{F}$ be a subspace of $C(\mathbf{J}, X)$. Then $\mathcal{F}$ is said to be *translation invariant* if $R_s\mathcal{F} \subset \mathcal{F}$ for all $s \in \mathbf{J}$. If $\varphi \in C(\mathbf{R}, X)$ then $L(\varphi)$ will stand for the subspace of $C(\mathbf{R}, X)$ generated by all translates of φ. We denote by

$$(1.1) \qquad \mathcal{PAP}_0(\mathbf{J}, X) = \{\varphi \in C(\mathbf{J}, X) : \lim_{t\to\infty} \frac{1}{t-c} \int_c^t \|\varphi(s)\|\, ds = 0\},$$

and

$$(1.2) \qquad \mathcal{E}(\mathbf{J}, X) = \{\varphi \in C_u(\mathbf{J}, X) : \lim_{t\to\infty} \frac{1}{t-c} \int_c^t \varphi(s+x)\, ds = M(\varphi)\},$$

where $c = 0$ if $\mathbf{J} = \mathbf{R}^+$ and $c = -t$ if $\mathbf{J} = \mathbf{R}$, and the limit in (1.2) exists uniformly with respect to $x \in \mathbf{J}$. The set $\mathcal{PAP}_0(\mathbf{J}, X)$ is a translation invariant, closed subspace of $C(\mathbf{J}, X)$ which does not contain any constant functions except 0. $\mathcal{PAP}_0(\mathbf{J}, X)$ is introduced

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by Zhang [26–28] and its elements are called *ergodic perturbations*. We remark that the elements of $\mathcal{PAP}_0(\mathbb{J},X)$ are not necessary uniformly continuous. $\mathcal{E}(\mathbb{J},X)$ is called the *space of ergodic functions* [2, 13, 18, 22] and $M(\varphi)$ is called the *mean* of φ.

Let $\varphi \in C(\mathbb{J},X)$. The function φ is called (weakly) almost periodic if the set $\{R_s\varphi : s \in \mathbb{J}\}$ is (weakly) relatively compact in $C(\mathbb{J},X)$. Denote by $\big(\mathcal{WAP}(\mathbb{J},X)\big)$ $\mathcal{AAP}(\mathbb{J},X)$ ($\mathbb{J} = \mathbb{R}^+$) all such functions. If $\mathbb{J} = \mathbb{R}$, then the space of almost periodic functions will be denoted by $\mathcal{AP}(\mathbb{R},X)$ (see [1], [11], [17]). Recently, Ruess and Summers [22, Proposition 2.1, Theorem 2.4] and Basit [2, Theorem 2.3.4 and 2.4.7] proved that

$$\mathcal{WAP}(\mathbb{J},X) \subset \mathcal{E}(\mathbb{J},X). \tag{1.3}$$

We recall that a subset E of $\mathbb{R}$ is called relatively dense in $\mathbb{R}$ if there exists a finite subset $\{t_1, t_2, \cdots, t_m\} \subset \mathbb{R}$ such that $\mathbb{R} = \bigcup_{i=1}^{m}(t_i + E)$. This implies that there exists $l > 0$ such that $]x, x+l[\cap E \neq \emptyset$ for all $x \in \mathbb{R}$. A function $\varphi \in C_u(\mathbb{R},X)$ is said to be recurrent if, for each $\epsilon > 0$ and $N > 0$ the set

$$E(\epsilon, N, \varphi) = \{\tau \in \mathbb{R} : \|\varphi(t+\tau) - \varphi(t)\| < \epsilon, |t| \leq N\}$$

is a relatively dense subset in $\mathbb{R}$. The set of all the recurrent functions will be denoted by $r(\mathbb{R},X)$.

It is known that $r(\mathbb{R},X)$ is not linear [14, 17]. Nevertheless, $\mathcal{AP}(\mathbb{R},X)$ is a linear class of $r(\mathbb{R},X)$ and also, $L(\varphi)$ is a linear class of $r(\mathbb{R},X)$ for all $\varphi \in r(\mathbb{R},X)$.

A function $\varphi \in C_u(\mathbb{R},X)$ is said to be almost automorphic if it is recurrent and for each $\epsilon > 0$ and $N > 0$ there exist $\delta > 0$ and $M > 0$ and a relatively dense subset $B(\delta, M, \varphi)$ such that

$$B(\delta, M, \varphi) - B(\delta, M, \varphi) \subset E(\epsilon, N, \varphi).$$

The set of all almost automorphic functions will be denoted by $\mathcal{AA}(\mathbb{R},X)$.

The following needed in the sequel statement is obvious.

PROPOSITION 1.1. *Let $\varphi \in \mathcal{AA}(\mathbb{R},X)$ and $\psi \in C_u(\mathbb{R},X)$. Let for each $\epsilon > 0$ and $N > 0$ there exist $\delta > 0$ and $M > 0$ such that*

$$E(\delta, M, \varphi) \subset E(\epsilon, N, \psi).$$

Then $\psi \in \mathcal{AA}(\mathbb{R},X)$.

If the range of φ is precompact, the above definition is equivalent to the definition which S. Bochner [10] introduced for numerical almost automorphic functions. The investigation of Veech [24] and Basit [6] (see also references in [2]) showed the relationship between almost automorphic functions and almost periodic functions in the sense of Levitan. The extension to vector-valued functions and many equivalent definitions are obtained in [6, 19, 24, 25].

We shall denote by $\mathcal{WAP}_0(\mathbb{J},X) = \{\varphi \in \mathcal{WAP}(\mathbb{J},X) : M(\|\varphi(\cdot)\|) = 0\}$, and $C_0(\mathbb{J},X) = \{\varphi \in C(\mathbb{J},X) : \lim_{|t|\to\infty} \varphi(t) = 0\}$. It is obvious that $C_0(\mathbb{J},X) \subset \mathcal{WAP}_0(\mathbb{J},X)$.

Let

(1.4) $\Pi_0(\mathbb{J},X) \in \{\mathcal{PAP}_0(\mathbb{J},X), C_0(\mathbb{J},X), \mathcal{WAP}_0(\mathbb{J},X)\}$.

$\Pi_c(\mathbb{R},X)$ denotes any class of functions satisfying:

(1.5) $\Pi_c(\mathbb{R},X)$ is a translation invariant, closed subspace of $C_u(\mathbb{R},X)$ containing all the constant functions.

(1.6) The map $m: \Pi_c(\mathbb{R},X) \longrightarrow \Pi_c(\mathbb{R}^+,X)$ defined by $m(\varphi) = \varphi|_{\mathbb{R}^+}$ is an isometry.

(1.7) $\Pi_c(\mathbb{R},X)$ is closed under multiplication by characters.

$\Pi_r(\mathbb{R},X)$ denotes any class of functions satisfying (1.5), (1.7) and

$$\Pi_r(\mathbb{R},X) \subset r(\mathbb{R},X). \tag{1.8}$$

It follows from [2, Proposition 2.1.7] that $\Pi_r(\mathbb{R},X)$ satisfies (1.6) and hence

$$\Pi_r(\mathbb{R},X) \subset \Pi_c(\mathbb{R},X) \text{ and } \Pi_r(\mathbb{R}^+,X) \subset \Pi_c(\mathbb{R}^+,X). \tag{1.9}$$

Define

$$\begin{aligned} \mathcal{A}\Pi_c(\mathbb{J},X) &= \Pi_c(\mathbb{J},X) + \Pi_0(\mathbb{J},X), \\ \mathcal{A}\Pi_r(\mathbb{J},X) &= \Pi_r(\mathbb{J},X) + \Pi_0(\mathbb{J},X). \end{aligned} \tag{1.10}$$

We have

$$\mathcal{A}\Pi_r(\mathbb{J},X) \subset \mathcal{A}\Pi_c(\mathbb{J},X). \tag{1.11}$$

If $\varphi = \psi + \xi$ with $\psi \in \Pi_c(\mathbb{J},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$, then ψ will be called the almost periodic type part and ξ the ergodic perturbation. We write

$$P_0\mathcal{A}\Pi_c(\mathbb{J},X), \quad C_0\mathcal{A}\Pi_c(\mathbb{J},X), \quad W_0\mathcal{A}\Pi_c(\mathbb{J},X)$$

instead of $\mathcal{A}\Pi_c(\mathbb{J},X)$ if

$$\Pi_0(\mathbb{J},X) = \mathcal{PAP}_0(\mathbb{J},X), \quad C_0(\mathbb{J},X), \quad \mathcal{WAP}_0(\mathbb{J},X)$$

respectively.

The space $\mathcal{A}\Pi_r(\mathbb{J},X)$ is a generalization of some spaces of almost periodic type functions. Here, we mention some of them:

$\mathcal{AAP}(\mathbb{J},X)$, the space of asymptotically almost periodic functions, $\mathcal{WAP}(\mathbb{J},X)$, the space of weakly almost periodic functions (see [2, 13, 15, 19, 22]); $\mathcal{PAP}(\mathbb{J},X)$, the space of pseudo almost periodic functions (see [26–28]); $\mathcal{AAA}(\mathbb{J},X)$, the space of asymptotically almost automorphic functions (see [2]). For other classes of functions satisfying (1.5)–(1.8) see [2]. It follows easily that $\Pi_c(\mathbb{R},X)$ are Λ-classes introduced in [2, Definition 2.2.1]. We prove that $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ is a Λ-class.

In this paper, we mainly study the space $\mathcal{A}\Pi_r(\mathbb{J},X)$. We prove that these classes are closed subspaces of $C(\mathbb{J},X)$ in section 2. Some recent results on indefinite integral obtained in [2] and [27] are extended in section 3. In section 4, we study the harmonic analysis introduced in [2] for these classes and give some of its application to ordinary differential equations. Since the elements of these classes are not necessary uniformly continuous, the application of the results of [2] is not direct. In Sections 5, we will investigate a specific $\mathcal{A}\Pi_r(\mathbb{J},X)$ space: pseudo and asymptotically almost automorphic functions and their solutions of quasilinear ordinary differential equations. We study the solutions of non-linear parabolic partial differential equations in section 6.

2. Properties of the Spaces $\mathcal{A}\Pi_c(\mathbb{J},X)$ and $\mathcal{A}\Pi_r(\mathbb{J},X)$.

In this section we study $\Pi_0(\mathbb{J},X)$, $\mathcal{A}\Pi_c(\mathbb{J},X)$ and $\mathcal{A}\Pi_r(\mathbb{J},X)$. We prove that every $\mathcal{A}\Pi_r(\mathbb{J},X)$ class is a closed subspace of $C(\mathbb{J},X)$ and every $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ class is a closed subspace of $C_u(\mathbb{J},X)$.

PROPOSITION 2.1. *If $\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)$ $\big(\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ and $\psi \in C(\mathbb{R},X)$ such that $\psi|_{\mathbb{J}} = \varphi$, then*

(i) $R_s\psi|_{\mathbb{J}} \in \mathcal{A}\Pi_c(\mathbb{J},X)\big(\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ for all $s \in \mathbb{R}$;

(ii) $\psi * f|_{\mathbb{J}} \in \mathcal{A}\Pi_c(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ $\big(\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)\big)$ for all $f \in L^1(\mathbb{R})$.

PROOF. First, we prove the case $\varphi \in \Pi_0(\mathbb{J},X)$.

(i) If $\mathbb{J} = \mathbb{R}$ then $\psi = \varphi$ and (i) follows from the fact that $\Pi_0(\mathbb{R},X)$ is translation invariant subspace of $C(\mathbb{R},X)$. If $\mathbb{J} = \mathbb{R}^+$, one can choose $\psi_0 \in C(\mathbb{R},X)$ such that $\psi_0(t) = \varphi(t)$ for $t \geq 0$, $\psi_0(t) = \varphi(0)(t+1)$ for $-1 \leq t < 0$, and $\psi_0(t) = 0$ for $t < -1$. A direct verification shows that $R_s\psi_0|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$ for all $s \in \mathbb{R}$. Since $R_s(\psi - \psi_0)|_{\mathbb{J}}$ has compact support for all $s \in \mathbb{R}$, we conclude that $R_s(\psi - \psi_0)|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$. This implies that $R_s\psi|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$.

(ii) If $h > 0$ and $f = \chi_{[0,h]}$ the characteristic function of $[0,h]$, then $\psi * f(t) = \int_0^h \psi(t+x)\,dx$. Direct computation for each of the three cases of $\Pi_0(\mathbb{J},X)$ shows that $\psi * f|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$. If $h < 0$, then $\int_0^h \psi(t+x)\,dx = \int_{-h}^0 \psi(t+x+h)\,dx = -\int_0^{-h} R_h\psi(t+x)\,dx$. By (i), we have that $R_h\psi|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$. Since $-h > 0$, it follows that $\int_0^{-h} R_h\psi(t+x)\,dx|_{\mathbb{J}}$ belongs to $\Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$. For $d > c$, define a function $g_{c,d}$ on $\mathbb{R}$ such that $g_{c,d}(t) = 1$ for $t \in [c,d]$ and $g_{c,d}(t) = 0$ otherwise. We have that

$$\psi * g_{c,d}(t) = \int_0^d \psi(t+x)\,dx - \int_0^c \psi(t+x)\,dx \quad (t \in \mathbb{R}).$$

Hence, we get $\psi * g_{c,d}|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ for all $c, d \in \mathbb{R}$. If $H(t) = \sum_{k=1}^m a_k g_{c_k,d_k}(t)$ is a step function, then, by the linearity of $\Pi_0(\mathbb{J},X)$ we get $H * \psi|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$. Since $f \in L^1(\mathbb{R})$, there exists a sequence of step functions $\{H_n\}$ such that $\|H_n - f\|_{L^1} \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that $\|\psi * H_n - \psi * f\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $\|\psi * H_n|_{\mathbb{J}} - \psi * f|_{\mathbb{J}}\| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $\psi * f|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$.

Secondly, let $\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)$, $\varphi = \zeta|_{\mathbb{J}} + \xi$, where $\zeta \in \Pi_c(\mathbb{R},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. The case $\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)$ can be proved exactly in the same way. Set $\eta = \psi - \zeta$. Then $\eta|_{\mathbb{J}} = \xi \in \Pi_0(\mathbb{J},X)$. By the first part (i), we get $R_s\eta|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$ and by property (1.5) of $\Pi_c(\mathbb{R},X)$, we have $R_s\zeta \in \Pi_c(\mathbb{R},X)$. This implies that $R_s\psi|_{\mathbb{J}} = R_s\zeta|_{\mathbb{J}} + R_s\eta|_{\mathbb{J}} \in \mathcal{A}\Pi_c(\mathbb{J},X)$. By the above $\eta * f|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ and from the inclusion $\zeta * f \in L(\zeta) \subset \Pi_c(\mathbb{R},X)$ for all $f \in L^1(\mathbb{R})$ by [2, Lemma 1.2.1], the proof of (ii) is complete.

COROLLARY 2.2. *Let* $\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)\big(\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ *and* $P\varphi(t) = \int_0^t \varphi(x)\,dx$. *Then* $\Delta_h P\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)\cap C_u(\mathbb{J},X)\big(\mathcal{A}\Pi_r(\mathbb{J},X)\cap C_u(\mathbb{J},X)\big)$ *for all* $h \in \mathbb{J}$.

PROPOSITION 2.3. $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ $\big(C_0\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ *is a closed subspace of* $C_u(\mathbb{J},X)$. *Moreover,* $C_0\mathcal{A}\Pi_c(\mathbb{J},X) = \Pi_c(\mathbb{J},X)\oplus C_0(\mathbb{J},X)$ $\big(C_0\mathcal{A}\Pi_r(\mathbb{J},X) = \Pi_r(\mathbb{J},X)\oplus C_0(\mathbb{J},X)\big)$.

PROOF. It is sufficient to prove the case $\varphi \in C_0\mathcal{A}\Pi_c(\mathbb{J},X)$. Let $\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_c(\mathbb{R},X)$ and $\xi \in C_0(\mathbb{J},X)$. We show that

$$\|\psi\| \le \|\varphi\|. \tag{2.1}$$

Indeed, $\|\varphi\| = \|\psi|_{\mathbb{J}} + \xi\| \ge \|R_s\psi|_{\mathbb{J}} + R_s\xi\| \ge \|R_s\psi|_{\mathbb{J}}\| - \|R_s\xi\|$ for $s \in \mathbb{J}$. Taking the limit when $s \longrightarrow \infty$ and using (1.6) and the fact $\lim_{s\to\infty}\|R_s\xi\| = 0$, we get (2.1).

Now, let $\{\varphi_n\}$ be Cauchy in $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$. Then $\varphi_n = \psi_n|_{\mathbb{J}} + \xi_n$, where $\psi_n \in \Pi_c(\mathbb{R},X)$ and $\xi_n \in C_0(\mathbb{J},X)$. It follows from (2.1) that $\{\psi_n\}$ is Cauchy too. By property (1.5), we conclude that $\{\psi_n\}$ converges to $\psi \in \Pi_c(\mathbb{R},X)$. This implies that $\{\xi_n\}$ is a Cauchy sequence in $C_0(\mathbb{R},X)$ and hence converges to $\xi \in C_0(\mathbb{J},X)$. Hence, $\{\varphi_n\}$ converges to $\psi|_{\mathbb{J}} + \xi \in C_0\mathcal{A}\Pi_c(\mathbb{J},X)$. Therefore, $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ is a closed subspace of $C_u(\mathbb{J},X)$.

Finally, we show that $C_0\mathcal{A}\Pi_c(\mathbb{J},X) = \Pi_c(\mathbb{J},X)\oplus C_0(\mathbb{J},X)$. Assuming that for φ from $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$, there are $\psi_i \in \Pi_c(\mathbb{R},X)$ and $\xi_i \in C_0(\mathbb{J},X)$, $i = 1,2$, such that $\varphi = \psi_i|_{\mathbb{J}} + \xi_i$. Then $0 = (\psi_1 - \psi_2)|_{\mathbb{J}} + (\xi_1 - \xi_2)$. By (2.1), $\|\psi_1 - \psi_2\| \le 0$. Therefore, $\psi_1 = \psi_2$ and $\xi_1 = \xi_2$. The proof is complete

We do not know whether or not $P_0\mathcal{A}\Pi_c(\mathbb{J},X)$ and $W_0\mathcal{A}\Pi_c(\mathbb{J},X)$ are closed subspaces of $C(\mathbb{J},X)$.

PROPOSITION 2.4. $\mathcal{A}\Pi_r(\mathbb{J},X)$ *is a closed subspace of* $C(\mathbb{J},X)$. *Moreover,* $\mathcal{A}\Pi_r(\mathbb{J},X) = \Pi_r(\mathbb{J},X)\oplus\Pi_0(\mathbb{J},X)$.

PROOF. By Proposition 2.3 the statement holds for the case that $\Pi_0(\mathbb{J},X) = C_0(\mathbb{J},X)$. Now, let $\Pi_0(\mathbb{J},X) = \mathcal{PAP}_0(\mathbb{J},X)$, let $\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \mathcal{PAP}_0(\mathbb{J},X)$. We show that

$$\psi(\mathbb{R}) \subset \overline{\varphi(\mathbb{J})}, \tag{2.2}$$

where $\psi(\mathbb{R})$ denotes the range of ψ and $\overline{\varphi(\mathbb{J})}$ denotes the closure in X of the range of φ. Assume that (2.2) is false. This implies there exists $t_0 \in \mathbb{R}$ such that $\psi(t_0) \notin \overline{\varphi(\mathbb{J})}$. Since ψ is recurrent, we can assume that $t_0 \in \mathbb{J}$. There exists an $\epsilon > 0$ such that $\inf_{s\in\mathbb{J}}\|\psi(t_0) - \varphi(s)\| > 2\epsilon$. Since ψ is continuous, there exists $\delta > 0$ such that

$$\inf_{s\in\mathbb{J}}\|R_{t_0}\psi(t) - R_{t_0}\varphi(s)\| > \epsilon \quad (|t| < \delta). \tag{2.3}$$

Since $R_{t_0}\psi$ is also recurrent, for $\epsilon > 0$ and $\delta > 0$ there exists $l_{\epsilon/2}$ such that each interval $(x, x + l_{\epsilon/2})$ contains a number τ with the property

$$\|R_{t_0}\psi(t+\tau) - R_{t_0}\psi(t)\| \le \frac{\epsilon}{2} \quad (|t| < \delta). \tag{2.4}$$

Let t and τ satisfy (2.4) and let $t+\tau \in \mathbf{J}$. It follows from (2.3) and (2.4) that

$$\begin{aligned}\|R_{t_0}\xi(t+\tau)\| &= \|R_{t_0}\varphi(t+\tau) - R_{t_0}\psi(t+\tau)\| \\ &\geq \|R_{t_0}\varphi(t+\tau) - R_{t_0}\psi(t)\| - \|R_{t_0}\psi(t) - R_{t_0}\psi(t+\tau)\| \\ &\geq \inf_{\tau}\|R_{t_0}\varphi(t+\tau) - R_{t_0}\psi(t)\| - \sup_{\tau}\|R_{t_0}\psi(t) - R_{t_0}\psi(t+\tau)\| \\ &\geq \frac{\epsilon}{2}.\end{aligned}$$

Since each interval $(x, x+l_{\epsilon/2})$ contains a number τ, it follows from inequality above that

$$\lim_{t\to\infty}\frac{1}{t-c}\int_c^t \|\xi(s)\|\,ds \geq \frac{\epsilon\delta}{2l_{\epsilon/2}}.$$

This contradicts the fact that $\xi \in \mathcal{PAP}_0(\mathbf{J},X)$. We have proved (2.2). It follows that $\|\psi\| \leq \|\varphi\|$.

Now, we can proceed exactly as in Proposition 2.3 to show that $\mathcal{A}\Pi_r(\mathbf{J},X)$ is a closed subspace of $C(\mathbf{J},X)$ and $\mathcal{A}\Pi_r(\mathbf{J},X) = \Pi_r(\mathbf{J},X) \oplus \mathcal{PAP}_0(\mathbf{J},X)$.

To show the assertion for the case that $\Pi_0(\mathbf{J},X) = \mathcal{WAP}_0(\mathbf{J},X)$, let $\varphi = \psi|_{\mathbf{J}} + \xi$, where $\psi \in \Pi_r(\mathbf{R},X)$ and $\xi \in \mathcal{WAP}_0(\mathbf{J},X)$. We need only to show that $\|\psi\| \leq \|\varphi\|$. For $x^* \in X^*$, the composition function $x^* \circ \xi$ is in $\mathcal{WAP}_0(\mathbf{J})$. Since $\mathcal{WAP}_0(\mathbf{J}) \subset \mathcal{PAP}_0(\mathbf{J})$ [9, 4.3.13], $x^* \circ \xi \in \mathcal{PAP}_0(\mathbf{J})$. Note that the numerical function $x^* \circ \psi$ is recurrent and $x^* \circ \varphi = x^* \circ \psi + x^* \circ \xi$, it follows that

$$|x^* \circ \psi|_\infty \leq |x^* \circ \varphi|_\infty \leq \|x^*\||\varphi|_\infty \quad (x^* \in X^*).$$

Therefore, $\|\psi\| \leq \|\varphi\|$. The proof is complete.

We remark that the above proposition gives a new proof that $\mathcal{WAP}(\mathbf{J},X) = \mathcal{AP}(\mathbf{J},X) \oplus \mathcal{WAP}_0(\mathbf{J},X)$ [15]; and $\mathcal{AAP}(\mathbf{J},X) = \mathcal{AP}(\mathbf{J},X) \oplus C_0(\mathbf{J},X)$ [18, 19]. As a consequence, we have

COROLLARY 2.5. *The spaces* $\mathcal{A}\Pi_r(\mathbf{J},X) \cap C_u(\mathbf{J},X), C_0\mathcal{A}\Pi_c(\mathbf{J},X)$ *are* Λ*-classes.*

Now, we prove

THEOREM 2.6. *Let* $\psi \in C_u(\mathbf{R},X)$. *Then* $\psi|_{\mathbf{J}} \in \mathcal{A}\Pi_r(\mathbf{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbf{J},X)\big)$ *if and only if* $\psi * f|_{\mathbf{J}} \in \mathcal{A}\Pi_r(\mathbf{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbf{J},X)\big)$ *for all* $f \in L^1(\mathbf{R})$.

PROOF. Necessity follows from Proposition 2.1. We show sufficiency. Let $h_n(t) = n/2$ for $|t| \leq 1/n$ and $h_n(t) = 0$ otherwise. Since $\psi \in C_u(\mathbf{R},X)$, we conclude that $h_n * \psi(t) = n/2\int_{-1/n}^{1/n}\psi(t+x)\,dx \to \psi(t)$ as $n \to \infty$ in $C_u(\mathbf{R},X)$. Hence $\|h_n * \psi|_{\mathbf{J}} - \psi|_{\mathbf{J}}\| \to 0$ as $n \to \infty$. By assumption, we have $h_n * \psi|_{\mathbf{J}} \in \mathcal{A}\Pi_r(\mathbf{J},X), n \in \mathbf{N}$. By Proposition 2.4, $\mathcal{A}\Pi_r(\mathbf{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbf{J},X)\big)$ is closed and hence the limit $\lim_{n\to\infty} h_n * \psi|_{\mathbf{J}} = \psi|_{\mathbf{J}} \in \mathcal{A}\Pi_r(\mathbf{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbf{J},X)\big)$.

3. Indefinite Integral of Functions in $\mathcal{A}\Pi_r(\mathbb{J},X)$.

In this section we study the indefinite integral of functions in $\mathcal{A}\Pi_r(\mathbb{J},X)$ $\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$. Throughout the section, if $\varphi \in C(\mathbb{J},X)$ then $P\varphi$ will denote the definite integral $P\varphi(t) = \int_0^t \varphi(x)\,dx$. We extend some results obtained in [2, 7, 16, 27].

THEOREM 3.1. *Let $\varphi \in \Pi_0(\mathbb{J},X)$. Then $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ if and only if there exists a constant $k \in X$ such that $P\varphi - k \in \Pi_0(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$.*

PROOF. We need only to prove the necessity. Let $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)$. Then $P\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. We have

$$\Delta_h P\varphi(t) = P\varphi(t+h) - P\varphi(t) = \int_0^h \varphi(t+x)\,dx$$

and

$$\Delta_h P\varphi(t) = \Delta_h\psi(t)|_{\mathbb{J}} + \Delta_h\xi(t) = [\psi(t+h) - \psi(t)]|_{\mathbb{J}} + [\xi(t+h) - \xi(t)].$$

By Corollary 2.2, we get $\Delta_h P\varphi \in \Pi_0(\mathbb{J},X)$ for all $h \in \mathbb{J}$. Since $\Pi_r(\mathbb{R},X)$ and $\Pi_0(\mathbb{J},X)$ are translation invariant, $\Delta_h\psi|_{\mathbb{J}} \in \Pi_r(\mathbb{J},X)$ and $\Delta_h\xi \in \Pi_0(\mathbb{J},X)$. By the uniqueness of the decomposition of elements of $\mathcal{A}\Pi_r(\mathbb{J},X)$, we get $\Delta_h\psi = 0$ and $\Delta_h\xi(t) = \int_0^h \varphi(t+x)\,dx$. Since ψ is continuous, we conclude that $\psi = k \in X$.

The case $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ can be proved in the same way.

We remark that Theorem 3.1 gives a new proof of [27, Theorem 7] first proved by Zhang for the case that $\Pi_r(\mathbb{R},X) = \mathcal{AP}(\mathbb{R},X)$ and $\Pi_0(\mathbb{J},X) = \mathcal{PAP}_0(\mathbb{J},X)$. As a consequence of Theorem 3.1, we have

COROLLARY 3.2. *Let $\varphi = \psi|_{\mathbb{J}} + \xi \in \mathcal{A}\Pi_r(\mathbb{J},X)$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. Then $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)$ if and only if $P\psi|_{\mathbb{J}} \in \Pi_r(\mathbb{J},X)$ and there exists $k \in X$ such that $P\xi - k \in \Pi_0(\mathbb{J},X)$.*

PROOF. We need only to prove the necessity. Let $P\varphi = \zeta|_{\mathbb{J}} + \eta$, where $\zeta \in \Pi_r(\mathbb{R},X)$ and $\eta \in \Pi_0(\mathbb{J},X)$. Then, for $h \in \mathbb{J}$,

$$\begin{aligned}\Delta_h P\varphi(t) &= \Delta_h\zeta|_{\mathbb{J}}(t) + \Delta_h\eta(t)\\ &= \Delta_h P\psi|_{\mathbb{J}}(t) + \Delta_h P\xi(t)\\ &= \int_0^h \psi(t+x)\,dx + \int_0^h \xi(t+x)\,dx.\end{aligned}$$

Similarly, we have that $\Delta_h\zeta|_{\mathbb{J}}, \int_0^h \psi(t+x)\,dx \in \Pi_r(\mathbb{J},X)$ and $\Delta_h\eta, \int_0^h \xi(t+x)\,dx \in \Pi_0(\mathbb{J},X)$. By the uniqueness of the decomposition, we get $\Delta_h(\eta - P\xi) = 0$. This implies that there exists $k \in X$ such that $P\xi = \eta + k$. Hence $P\psi|_{\mathbb{J}} = \zeta|_{\mathbb{J}} - k$ and $P\psi|_{\mathbb{J}} \in \Pi_r(\mathbb{J},X)$. Corollary 3.2 holds also for the case $\varphi \in C_0\mathcal{A}\Pi_c(\mathbb{J},X)$.

Now, we study sufficient conditions for $P\varphi$ in $\mathcal{A}\Pi_r(\mathbb{J},X)$. We have

THEOREM 3.3. *Let $\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ and $P\varphi \in \mathcal{E}(\mathbb{J},X)$. Then $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$.*

PROOF. Let $\varphi = \psi + \xi$ with $\psi \in \Pi_r(\mathbb{J},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. By Corollary 2.2, $\Delta_h P\xi \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ and by [2, Lemma 1.2.1], $\Delta_h P\psi \in L(\varphi) \subset \Pi_r(\mathbb{R},X)$ for all $h \in \mathbb{R}$.

This implies that $\Delta_h P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},x)$. By Corollary 2.5 $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{R},X)$ is a Λ-class. Since $P\varphi \in \mathcal{E}(\mathbb{R},X), P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ by [2, Theorem 3.1.2]. By Corollary 2.5, $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ is a Λ-class, hence the statement follows directly from [2, Theorem 3.1.1].

4. **Harmonic Analysis of $\mathcal{A}\Pi_r(\mathbb{J},X)$ and Solutions of Linear Differential Equations.** In this section we introduce the spectrum $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi)$ of the function $\varphi \in C(\mathbb{R},X)$ with respective to $\mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$. This kind of spectrum proved to be a useful tool to examine solutions of differential equations of many function classes [2, 5, 8, 17 Ch.6]. See also the recent works [4], [20], [23]. We apply the results on spectrum to the differences of functions and linear differential equations on the half line. Since the proofs for the case $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ are the same as in the case $C_0\mathcal{A}\Pi_r(\mathbb{J},X)$ we restrict ourselves to $\mathcal{A}\Pi_r(\mathbb{J},X)$.

Let $\varphi \in C(\mathbb{R},X)$. Denote by $I_{\mathcal{A}\Pi_r}(\varphi) = \{f \in L^1(\mathbb{R}) : f * \varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)\}$. Since $\mathcal{A}\Pi_r(\mathbb{J},X)$ is a closed subspace of $C(\mathbb{J},X), I_{\mathcal{A}\Pi_r}(\varphi)$ is a closed ideal of $L^1(\mathbb{R})$. We set

$$\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) = \text{ hull } I_{\mathcal{A}\Pi_r}(\varphi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0, f \in I_{\mathcal{A}\Pi_r}(\varphi)\}. \tag{4.1}$$

We denote by $\mathrm{sp}(\varphi)$ the Beurling spectrum of φ [21, p138]. Similar to the corresponding properties of Beurling spectrum, the following can be proved exactly as in [2].

PROPOSITION 4.1. *Let $\varphi \in C(\mathbb{R},X)$ and $f \in L^1(\mathbb{R})$. Then*

(a) $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) \subset \mathrm{sp}(\varphi)$*;*

(b) $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi * f) \subset \mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) \cap \mathrm{supp}\hat{f}; f \in L^1(\mathbb{R})$ *such that* $f \in I_{\mathcal{A}\Pi_r}(\varphi)$ *and* $\hat{f}(\lambda) \neq 0$.

(c) Let $\varphi \in C_u(\mathbb{R},X)$. Then $\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ if and only if $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) = \emptyset$.

THEOREM 4.2. *Let $\varphi \in C_u(\mathbb{R},X)$ and let $\mathrm{sp}(\varphi)$ be separated from zero.*

(a) If $\Delta_h\varphi|_{\mathbb{J}} = (R_h\varphi - \varphi)|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ for all $h \in \mathbb{R}$, then $\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$.

(b) If $\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$, then $P\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$.

PROOF. Corollary 2.5, $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ is a Λ-class. Therefore (a) follows directly from [2, Theorem 4.2.4].

(b) By [3, Corollary 4.4] $P\varphi$ is a bounded uniformly continuous function and there exists $k \in X$ such that $\mathrm{sp}(P\varphi + k)$ is isolated from zero. By Corollary 2.2, $\Delta_h P\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ for all $h \in \mathbb{R}$. Therefore, $P\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ by (a).

Now we study the linear differential equation

$$y'' + a_1y' + a_0y = \varphi \quad \big(\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big). \tag{4.2}$$

We give sufficient conditions under which the bounded solutions of (4.2) is in $\mathcal{A}\Pi_r(\mathbb{J},X)$.

THEOREM 4.3. *If $a_0, a_1 \in \mathbb{C}$ in (4.2) are such that $(i\alpha)^2 + i\alpha a_1 + a_0 \neq 0$ for all $\alpha \in \mathbb{R}$ and if y is a bounded solution of (4.2), then $y \in \mathcal{A}\Pi_r(\mathbb{J},X)$.*

PROOF. If $\mathbb{J} = \mathbb{R}^+$, we define $Y(t) = y(t)$ for $t \geq 0$ and $Y(t) = y(0)\cos(1 - \cos t) + y'(0)\sin t + y''(0)(\sin^2 t)/2$ for $t < 0$. It is easy to verify that Y is a solution of the differential equation:

$$Y'' + a_1Y' + a_0Y = \psi \tag{4.3}$$

where $\psi(t) = \varphi(t)$ for $t \geq 0$ and $\psi(t) = Y'' + a_1 Y' + a_0 Y$ for $t < 0$. Since Y is bounded, by Landau-Esclangon lemma we conclude that Y', Y'' are also bounded (see [17, Proposition 4, p.95]). This implies that Y is uniformly continuous. It is easy to check that $\psi \in C(\mathbb{R}, X)$. We prove that $\mathrm{sp}_{\mathcal{A}\Pi_r}(Y) = \emptyset$. Indeed, let $\alpha \in \mathbb{R}$, there exists $f \in S(\mathbb{R})$ such that $\hat{f}(\alpha) \neq 0$. We have $f * Y'' + f * a_1 Y' + f * a_0 Y = f * \psi$. Hence $Y * [f'' + a_1 f' + a_0 f] = \psi * f$. We have $(f'' + \widehat{a_1 f'} + a_0 f)(\alpha) = [(i\alpha)^2 + i\alpha a_1 + a_0]\hat{f}(\alpha)$. From the assumptions we conclude that the function $g := f'' + a_1 f' + a_0 f$ satisfies $\hat{g}(\alpha) \neq 0$. From (4.1), it follows that $\alpha \notin \mathrm{sp}_{\mathcal{A}\Pi_r}(Y)$. Since α is arbitrary, we get $\mathrm{sp}_{\mathcal{A}\Pi_r}(Y) = \emptyset$. By Proposition 4.1 (c), $y = Y|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J}, X)$.

In the same way, we have

COROLLARY 4.4. *Let $\varphi \in \mathcal{A}\Pi_r(\mathbb{J}, X)$. Let $\lambda = u + iv$ with $u \neq 0$. Then every bounded solution of the equation $y' = \lambda y + \varphi$ belongs to $\mathcal{A}\Pi_r(\mathbb{J}, X)$.*

THEOREM 4.5. *If $\lambda = u + iv \in \mathbb{C}$ with $u > 0$ and $\varphi \in \mathcal{A}\Pi_r(\mathbb{J}, X)$, then the solutions of the equation*

$$\frac{dy}{dx} = \lambda y + \varphi \tag{4.4}$$

of the form $y(x) = e^{\lambda x}[c + \int_0^x e^{-\lambda t}\varphi(t)\,dt]$ are bounded on $\mathbb{R}^+$. Furthermore, if $\mathbb{J} = \mathbb{R}^+$, then $y \in \mathcal{A}\Pi_r(\mathbb{R}^+, X)$ and $\|y\| \leq u^{-1}\|\varphi\|$. If $\mathbb{J} = \mathbb{R}$, then (4.4) has a unique bounded solution defined on $\mathbb{R}$ if and only if $y(0) = -\int_0^\infty e^{-\lambda t}\varphi(t)\,dt$. In this case the function $y_0(x) = -\int_x^\infty e^{\lambda(x-t)}\varphi(t)\,dt$ is a bounded solution from $\mathcal{A}\Pi_r(\mathbb{R}, X)$.

PROOF. Direct verification shows that

$$y(x) = e^{\lambda x}[c + \int_0^x e^{-\lambda t}\varphi(t)\,dt], \tag{4.5}$$

is a bounded solution of (4.4). If $\mathbb{J} = \mathbb{R}^+$, then $y \in \mathcal{A}\Pi_r(\mathbb{R}^+, X)$ by Corollary 4.4. Since $u > 0, |e^{\lambda x}| = e^{ux} \longrightarrow \infty$ as $x \longrightarrow \infty$. For y to be bounded on $\mathbb{R}$, we must have $c + \int_0^x e^{-\lambda t}\varphi(t)dt \longrightarrow 0$ as $x \longrightarrow \infty$. This means that we must take in (4.5) $c = -\int_0^\infty e^{-\lambda t}\varphi(t)\,dt$. We note that the improper integral in (4.5) is convergent since $|e^{-\lambda t}\varphi(t)| \leq \|\varphi\| e^{-ut}$ for $t \geq 0$. Thus, the unique bounded solution of equation (4.4) can only be $y_0(x) = -\int_x^\infty e^{\lambda(x-t)}\varphi(t)\,dt$, and we do have $\|y_0(x)\| \leq \|\varphi\| e^{ux} \int_x^\infty e^{-ut}\,dt = \|\varphi\|/u$, so that y_0 is bounded. By Corollary 4.4, $y_0 \in \mathcal{A}\Pi_r(\mathbb{R}, X)$.

When $u < 0$, we have that $y_0(x) = \int_{-\infty}^x e^{\lambda(t-x)}\varphi(t)\,dt$ is the unique bounded solution of (4.3) and $y_0 \in \mathcal{A}\Pi_r(\mathbb{R}, X)$. At the same time we have the similar estimate $\|y_0(x)\| \leq \|\varphi\|/|u|$.

THEOREM 4.6. *Let $\psi \in C(\mathbb{R}, X)$ and $\psi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J}, X)$. Let $\varphi(t) = e^{-ivt}\psi(t)$ with $v \in \mathbb{R}$ and let the Beurling spectrum of φ be separated from zero. Then all solutions of equation (4.3) belong to $\mathcal{A}\Pi_r(\mathbb{J}, X) \cap C_u(\mathbb{J}, X)$.*

PROOF. (4.4) has general solution $y(x) = e^{ivx}[c + \int_0^x e^{-ivt}\psi(t)\,dt]$, where c is an arbitrary element of X. By Theorem 4.2 (b), we conclude $\int_0^x e^{-ivt}\psi(t)\,dt]|_{\mathbb{J}}$ is in $\mathcal{A}\Pi_r(\mathbb{J}, X) \cap C_u(\mathbb{J}, X)$. Since $\mathcal{A}\Pi_r(\mathbb{J}, X)$ contains all constant functions and invariant under multiplication by characters, $y \in \mathcal{A}\Pi_r(\mathbb{J}, X) \cap C_u(\mathbb{J}, X)$.

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5. **Asymptotically and Pseudo Almost Automorphic Functions.** In this section we will concentrate two new, specific $\mathcal{A}\Pi_r(\mathbb{J},X)$ spaces. That is the space of asymptotically almost automorphic introduced in [2] and the space of pseudo almost automorphic functions defined below.

DEFINITION 5.1. A function $\psi \in \mathcal{C}(\mathbb{J},X)$ is called *pseudo (asymptotically) almost automorphic* if $\psi = \varphi|_{\mathbb{J}} + \xi$, where $\varphi \in \mathcal{AA}(\mathbb{R},X)$ and $\xi \in \mathcal{PAP}_0(\mathbb{J},X)\big(\mathcal{C}_0(\mathbb{J},X)\big)$. The functions φ and ξ are called the almost automorphic component and the ergodic perturbation respectively of the function ψ. Denote by $\mathcal{PAA}(\mathbb{J},X)\big(\mathcal{AAA}(\mathbb{J},X)\big)$ the set of all such functions ψ.

By Theorem 2.1.1 and Proposition 2.1.2 in [2], $\mathcal{AA}(\mathbb{R},X)$ satisfies (1.5)–(1.8). Therefore, $\mathcal{AA}(\mathbb{R},X)$ is a $\Pi_r(\mathbb{R},X)$ space and $\mathcal{PAA}(\mathbb{J},X)$ and $\mathcal{AAA}(\mathbb{J},X)$ are $\mathcal{A}\Pi_r(\mathbb{J},X)$ spaces. It follows from Proposition 2.4 that

$$\mathcal{PAA}(\mathbb{J},X) = \mathcal{AA}(\mathbb{R},X)|_{\mathbb{J}} \oplus \mathcal{PAP}_0(\mathbb{J},X),$$
$$\mathcal{AAA}(\mathbb{J},X) = \mathcal{AA}(\mathbb{R},X)|_{\mathbb{J}} \oplus \mathcal{C}_0(\mathbb{J},X)$$

and

$$\varphi(\mathbb{R}) \subset \overline{\psi(\mathbb{J})}, \quad \|\varphi\| \leq \|\psi\|. \tag{5.1}$$

THEOREM 5.2. *The following statements hold.*

(1) A function $\xi \in \mathcal{C}\mathbb{R}$ is in $\mathcal{PAP}_0(\mathbb{R})\big(\mathcal{C}_0(\mathbb{R})\big)$ if and only if ξ^2 is.

(2) $\Xi \in \mathcal{C}(\mathbb{R})^n$ is in $\mathcal{PAP}_0(\mathbb{R})^n\big(\mathcal{C}_0(\mathbb{R})^n\big)$ if and only if the norm function $|\Xi(\cdot)|$ is in $\mathcal{PAP}_0\mathbb{R}\big(\mathcal{C}_0(\mathbb{R})^n\big)$.

PROOF. We show the theorem only for the case of $\mathcal{PAP}_0(\mathbb{R})$. The case of $\mathcal{C}_0(\mathbb{R})$ is similar.

(1) The sufficiency follows since

$$\frac{1}{2t}\int_{-t}^{t} |\xi(x)|/\,dx \leq \frac{1}{2t}[\int_{-t}^{t} |\xi(x)|^2\,dx]^{1/2}[\int_{-t}^{t} 1\,dx]^{1/2} = [\frac{1}{2t}\int_{-t}^{t} |\xi(x)|^2\,dx]^{1/2}.$$

The necessity follows from the fact that $\mathcal{PAP}_0(\mathbb{R})$ is an ideal of $\mathcal{C}(\mathbb{R})$.

(2) By (1), $\Xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathcal{PAP}_0(\mathbb{R})^n$ if and only if $\xi_i\overline{\xi_i} \in \mathcal{PAP}_0(\mathbb{R}), i = 1, 2, \cdots, n$. The latter is equivalent to that $|\Xi(\cdot)|^2 = \sum_{i=1}^{n} |\xi_i(\cdot)|^2 \in \mathcal{PAP}_0(\mathbb{R})$, which, again by (1), is equivalent to that $|\Xi(\cdot)| \in \mathcal{PAP}_0(\mathbb{R})$.

Let $\Omega \subset \mathbb{C}^n$ be compact and define

$$\mathcal{PAP}_0(\Omega \times \mathbb{R}) = \mathcal{PAP}_0\big(\mathbb{R}, \mathcal{C}(\Omega)\big), \mathcal{C}_0(\Omega \times \mathbb{R}) = \mathcal{C}_0\big(\mathbb{R}, \mathcal{C}(\Omega)\big)$$

and

$$\mathcal{AA}(\Omega \times \mathbb{R}) = \mathcal{AA}\big(\mathbb{R}, \mathcal{C}(\Omega)\big).$$

A function $\varphi \in \mathcal{AA}(\Omega \times \mathbb{R})$ is called almost automorphic in $t \in \mathbb{R}$ uniformly in $Z \in \Omega$.

$\mathcal{PAA}(\Omega \times \mathbb{R})\big(\mathcal{AAA}(\Omega \times \mathbb{R})\big)$ is defined to consist of functions ψ such that

$$\psi = \varphi + \xi \quad \Big(\varphi \in \mathcal{AA}(\Omega \times \mathbb{R}), \xi \in \mathcal{PAP}_0(\Omega \times \mathbb{R})\big(\mathcal{C}_0(\Omega \times \mathbb{R})\big)\Big).$$

The following lemma is straight forward.

LEMMA 5.3. *Let* $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in C(\mathbb{R})^n$. *Then* $\Phi \in \mathcal{AA}(\mathbb{R}, \mathbb{C}^n)$ *if and only if* $\Phi \in \mathcal{AA}(\mathbb{R})^n$.

For $H = (h_1, h_2, \cdots, h_n) \in C(\mathbb{R})^n$, suppose that $H(t) \in \Omega$ for all $t \in \mathbb{R}$. Define

$$H \times \iota : \mathbb{R} \to \Omega \times \mathbb{R} \text{ by } H \times \iota(t) = \big(h_1(t), h_2(t), \cdots, h_n(t), t\big) \quad (t \in \mathbb{R}).$$

For $\Psi = (\psi_1, \psi_2, \cdots, \psi_n) \in \mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$, let $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ and $\Xi = (\xi_1, \xi_2, \cdots, \xi_n)$, where φ_i and ξ_i are the almost automorphic component and the ergodic perturbation respectively of ψ_i, $i = 1, 2, \cdots, n$.

The following theorem generalizes [26, Theorem 1.5]

THEOREM 5.4. *Let* $\psi \in \mathcal{PAA}\big(\mathbb{J}, C(\Omega)\big)\Big(\mathcal{AAA}\big(\mathbb{J}, C(\Omega)\big)\Big)$. *If* $\Psi \in \mathcal{PAA}(\mathbb{J})^n\big(\mathcal{AAA}(\mathbb{J})^n\big)$ *and* $\Psi(\mathbb{J}) \subset \Omega$, *then* $\psi \circ (\Psi \times \iota) \in \mathcal{PAA}(\mathbb{J})\big(\mathcal{AAA}(\mathbb{J})\big)$.

PROOF. It is sufficient to prove the case $\mathbb{J} = \mathbb{R}, \psi \in \mathcal{PAA}\big(\mathbb{J}, C(\Omega)\big)$, and $\Psi \in \mathcal{PAA}(\mathbb{J})^n$.

Let $\psi = \varphi + \xi$ and $\Psi = \Phi + \Xi$ with $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in \mathcal{AA}(\mathbb{R})^n$, as above. Since $\Psi(t) \in \Omega$ when $t \in \mathbb{R}$, it follows from (5.1) that $\Phi(t) \in \Omega$ for $t \in \mathbb{R}$. Note that

$$\psi\circ(\Psi\times\iota) = \varphi\circ(\Psi\times\iota)+\xi\circ(\Psi\times\iota) = \varphi\circ(\Phi\times\iota)+[\varphi\circ(\Psi\times\iota)-\varphi\circ(\Phi\times\iota)+\xi\circ(\Psi\times\iota)].$$

We show that $\varphi \circ (\Phi \times \iota) \in \mathcal{AA}(\mathbb{R})$. Let $\epsilon > 0$. Since $\varphi \in C_u(\Omega \times \mathbb{R})$, there is a $\delta, \epsilon/2 > \delta > 0$ such that

$$(5.2) \qquad |\varphi(Z_1, t) - \varphi(Z_2, t)| < \frac{\epsilon}{2} \quad (Z_1, Z_2 \in \Omega, |Z_1 - Z_2| < \delta; t \in \mathbb{R}).$$

Note Lemma 5.3 and the hypothesis that $\Phi \in \mathcal{AA}(\mathbb{R})^n$ and $\varphi \in \mathcal{AA}\big(\mathbb{R}, C(\Omega)\big)$, we conclude $(\varphi, \Phi) \in \mathcal{AA}\big(\mathbb{R}, C(\Omega) \times \mathbb{R}^n\big)$. From the identity

$$(5.3) \qquad \begin{aligned} \varphi\big(\Phi(t+\tau), t+\tau\big) - \varphi\big(\Phi(t), t\big) &= \varphi\big(\Phi(t+\tau), t+\tau\big) \\ &\quad - \varphi\big(\Phi(t), t+\tau\big) + \varphi\big(\Phi(t), t+\tau\big) - \varphi\big(\Phi(t), t\big) \end{aligned}$$

we conclude $E\big(\delta, N, (\varphi, \Phi)\big) \subset E\big(\delta, N, (\varphi \circ \Phi \times \iota)\big)$. This implies $\varphi \circ (\Phi \times \iota) \in \mathcal{AA}(\mathbb{R})$ by Proposition 1.1.

To finish the proof, we need to show that the function $h = \varphi \circ (\Psi \times \iota) - \varphi \circ (\Phi \times \iota) + \xi \circ (\Psi \times \iota)$ is in $\mathcal{PAP}_0(\mathbb{R})$. First we show that $\varphi \circ (\Psi \times \iota) - \varphi \circ (\Phi \times \iota) \in \mathcal{PAP}_0(\mathbb{R})$. It is trivial in the case that $\varphi = 0$. So we assume that $\varphi \not\equiv 0$. Let ϵ and δ be as in (5.2). Set

$$(5.4) \qquad C_\delta = \{t \in \mathbb{R} : |\Psi(t) - \Phi(t)| = |\Xi(t)| \geq \delta\}.$$

It follows from [27, Definition 3 and Proposition 4] that there is a $T > 0$ such that when $t \geq T$

$$(5.5) \qquad \frac{m([-t, t] \cap C_\delta)}{2t} < \frac{\epsilon}{4\|\varphi\|},$$

where m stands for the Lebesgue measure on $\mathbb{R}$. It follows from (5.2), (5.4) and (5.5) that

$$\begin{aligned}&\frac{1}{2t}\int_{-t}^{t}|\varphi\big(\Psi(s),s\big)-\varphi\big(\Phi(s),s\big)|\,ds\\&\quad=\frac{1}{2t}\left\{\int_{[-t,t]\setminus C_\delta}+\int_{[-t,t]\cap C_\delta}|\varphi\big(\Psi(s),s\big)-\varphi\big(\Phi(s),s\big)|\,ds\right\}\\&\quad\le\frac{\epsilon}{2}+2\|\varphi\|\frac{m([-t,t]\cap C_\delta)}{2t}<\epsilon.\end{aligned}$$

Therefore, $\varphi\circ(\Psi\times\iota)-\varphi\circ(\Phi\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$.

A little modification of the proof for [26, Theorem 1.5] shows that $\xi\circ(\Psi\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$. The proof is complete.

THEOREM 5.5. *Consider systems of the form*

$$\frac{dY}{dx}=AY+\Psi, \tag{5.6}$$

where $A=(a_{ij})$ *is a complex* $n\times n$ *matrix and* $\Psi=(\psi_1,\psi_2,\cdots,\psi_n)'\in\mathcal{PAA}(\mathbb{R})^n$ $\big(\mathcal{AAA}(\mathbb{R})^n\big)$. *If the matrix* $A=(a_{ij})$ *has no eigenvalues with real part zero, then system (5.6) admits a unique solution* $Y=(y_1,y_2,\cdots,y_n)'\in\mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$. *Moreover*

$$\|Y\|\le K\|\Psi\|, \tag{5.7}$$

where $K>0$ *depends only on the matrix* A.

PROOF. By a discussion in [11, Theorem 4.2] the matrix A can be considered triangular. Therefore the theorem follows by applying Theorem 4.5 n times.

Now, consider a system of the form

$$\frac{dY}{dx}=AY+\Psi+\mu G\circ(Y\times\iota), \tag{5.8}$$

where $\mu\in\mathbb{C}\setminus\{0\}$, A is a complex $n\times n$ matrix, $\Psi\in\mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$, and $G\in\mathcal{PAA}(\Omega\times\mathbb{R})^n\ \big(\mathcal{AAA}(\Omega\times\mathbb{R})^n\big)$. Such a system is called quasi-linear. We get the generating system of (5.8) by putting $\mu=0$.

As in the proof for [26, Theorem 2.3], by using Theorems 5.4 and 5.5 one shows the following theorem.

THEOREM 5.6. *Let* Ψ *and* A *be as in Theorem 5.5. Let* $Y^{(0)}$ *be the unique solution in* $\mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$ *of the generating system of (5.8), let* $a>0$ *and let* $\Omega=\bigcup\{Z\in\mathbb{C}^n:|Z-Y^{(0)}(t)|\le a,t\in\mathbb{R}\}$. *Assume that*

(1) $G\in\mathcal{PAA}(\Omega\times\mathbb{R})^n\ \big(\mathcal{AAA}(\Omega\times\mathbb{R})^n\big)$ *such that*

$$|G(Z',t)-G(Z'',t)|\le L\Sigma_{i=1}^{n}|z_i'-z_i''|,\quad (Z',Z''\in\Omega,t\in\mathbb{R}), \tag{5.9}$$

where $L>0$;

(2) $0<|\mu|<\min\{1/LK,a/K\|G\|\}$, *where* $K>0$ *in (5.7) depends only on the matrix* A.

Then there exists a unique solution $Y=(y_1,y_2,\cdots,y_n)'\in\mathcal{PAA}(\mathbb{R})^n\ \big(\mathcal{AAA}(\mathbb{R})^n\big)$ *of the system (5.8) such that* $Y(x)\in\Omega$ *for all* $x\in\mathbb{R}$. *Furthermore,* $\|Y-Y^{(0)}\|\longrightarrow 0$ *as* $\mu\longrightarrow 0$.

6. **The Solutions of non-linear Parabolic Partial Differential Equations.** For $T > 0$ and $A > 0$, let $\Delta = \mathbb{R} \times [0, T]$ and $\Delta_1 = \Delta \times [-A, A]$. Consider the non-linear parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \psi(x, t, u) \quad \big((x, t) \in \Delta\big), \tag{6.1}$$

where $\psi \in C(\Delta_1)$ satisfies the Lipschitz condition

$$|\psi(x, t, v') - \psi(x, t, v'')| \leq l|v' - v''| \quad \big((x, t) \in \Delta, v', v'' \in [-A, A]). \tag{6.2}$$

In this section, we shall use some knowledge from ordinary differential equations to establish some properties of solutions of (6.1). For this purpose, we first consider the following ordinary differential equation

$$\frac{d^2 y}{dx^2} - \alpha^2 y = r, \tag{6.3}$$

where $\alpha > 0$ and $r \in C(\mathbb{R})$. Note that (6.3) admits a unique bounded solution given by

$$y_0(x) = -\frac{1}{2\alpha}\left\{e^{\alpha x} \int_x^\infty e^{-\alpha t} r(t)\, dt + e^{-\alpha x} \int_{-\infty}^x e^{\alpha t} r(t)\, dt\right\}, \tag{6.4}$$

for which

$$\|y_0\| \leq \frac{1}{\alpha^2}\|r\|. \tag{6.5}$$

It follows from Theorem 4.3 that y_0 in (6.4) is in $\mathcal{PAA}(\mathbb{R})\big(\mathcal{AAA}(\mathbb{R})\big)$ if r is.

Let us now consider the system of ordinary differential equations

$$\frac{d^2 u_k}{dx^2} = h^{-1}[u_k - u_{k-1}] + \psi(x, t_k, u_{k-1}), \quad k = 1, 2, \cdots, n, \tag{6.6}$$

where $h = T/n \in \mathbb{R}$, $t_k = kh$ and $u_0(x) = u(x, 0)$, where u is a solution of (6.1). We need to consider (6.6) for variable n; note that the partition $\{t_k\}_{k=0}^n$ of $[0, T]$ depends on n, as do all the functions in a solution $(u_1, u_2, \cdots, u_n)$ of (6.6).

As in the proof for [26, Lemma 3.1], one shows the following lemma.

LEMMA 6.1. *Let* $\psi \in \mathcal{PAA}(\Delta_1)(\mathcal{AAA}(\Delta_1))$ *satisfy the Lipschitz condition (6.2). Suppose* u *is a solution of equation (6.1) such that* $\|u\| < A$, u *and* $\partial u/\partial t$ *are uniformly continuous, and* $u_0 = u(\cdot, 0) \in \mathcal{PAA}(\mathbb{R})\ \big(\mathcal{AAA}(\mathbb{R})\big)$. *Then there exist* $n_0 \in \mathbb{N}$ *and* $W \geq 0$ *such that (6.6) has a unique solution* $(u_1, u_2, \cdots, u_n) \in \mathcal{PAA}(\mathbb{R})^n\ \big(\mathcal{AAA}(\mathbb{R})^n\big)$ *for* $n \geq n_0$*; it satisfies*

(1) $\|u_k\| \leq A$, $1 \leq k \leq n$, *and*

(2) the functions $\varepsilon_0 = 0$, $\varepsilon_k = u(\cdot, t_k) - u_k$ *are in* $C(\mathbb{R})$ *with*

$$\|\varepsilon_k\| \leq W\omega(h), \tag{6.7}$$

where $h = T/n$, $t_k = kh$, and ω is the modulus of uniform continuity of u and $\partial u/\partial t$.

THEOREM 6.2. *Let ψ and u satisfy the conditions of Lemma 6.1. Then u is in $\mathcal{PAA}(\Delta)(\mathcal{AAA}(\Delta))$.*

PROOF. We only show the assertion that $u \in \mathcal{PAA}(\Delta)$. Similarly, one shows the assertion that $u \in \mathcal{AAA}(\Delta)$. Let n_0 be as in Lemma 6.1, and fixed $n \geq n_0$ and $t \in [0, T]$. Then there is a $k_0 \leq n$ such that $|t - t_{k_0}| < h$. Recall that $h = T/n$ and $t_k = kh$. If $B_n = \{u_k : k = 1, 2, \cdots, n\} \subset \mathcal{PAA}(\mathbf{R})$ is the solution of (6.6) given by Lemma 6.1, the uniform continuity of u gives

$$\begin{aligned} |u(x,t) - u_{k_0}(x)| &\leq |u(x,t) - u(x,t_{k_0})| + |u(x,t_{k_0}) - u_{k_0}(x)| \\ &< (1 + W)\omega(h) \quad (x \in \mathbf{R}). \end{aligned} \tag{6.8}$$

It follows that the function $u(\cdot, t)$ is in the norm closure of $\bigcup_{n=n_0}^{\infty} B_n$; hence $u(\cdot, t) \in \mathcal{PAA}(\mathbf{R})$ (Proposition 2.4).

Let $\varphi(\cdot, t)$ and $\xi(\cdot, t)$, φ_k and ξ_k be the almost automorphic components and the ergodic perturbations respectively of $u(\cdot, t)$, u_k, $k = 1, 2, \cdots, n$. To show that $u \in \mathcal{PAA}(\Delta)$, we need to prove that $\varphi \in \mathcal{AA}(\Delta)$ and $\xi \in \mathcal{PAP}_0(\Delta)$.

Since $u(\cdot, t'), u(\cdot, t'') \in \mathcal{PAA}(\mathbf{R})$ for any $t', t'' \in [0, T]$, so is $u(\cdot, t') - u(\cdot, t'')$ and also $u(\cdot, t) - u_k$ for $t \in [0, T]$ and $k = 1, 2, \cdots, n$. It follows from (5.1) that

$$\|\varphi(\cdot, t') - \varphi(\cdot, t'')\| \leq \|u(\cdot, t') - u(\cdot, t'')\| \tag{6.9}$$

and

$$\|\varphi(\cdot, t) - \varphi_k\| \leq \|u(\cdot, t) - u_k\| \quad k = 1, 2, \cdots, n. \tag{6.10}$$

We show that $\varphi \in \mathcal{AA}(\Delta)$. Let $\epsilon > 0$. Choose $n \geq n_0$ such that $h = T/n$ implies

$$4(W + 1)\omega(h) < \epsilon. \tag{6.11}$$

Let $\Phi = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$. By Lemma 5.3, $\Phi \in \mathcal{AA}(\mathbf{R}, \mathbf{C}^n)$. Let $N > 0$. It follows from (6.7), (6.9), (6.10) and (6.11) that, for $|x| < N, \tau \in E(\epsilon/2, N, \Phi)$ and $t \in [0, T]$, there exists a t_{k_0} such that $|t - t_{k_0}| < h$ and

$$\begin{aligned} |\varphi(x+\tau, t) - \varphi(x, t)| &\leq |\varphi(x+\tau, t) - \varphi(x+\tau, t_{k_0})| + |\varphi(x+\tau, t_{k_0}) - \varphi_{k_0}(x+\tau)| \\ &\quad + |\varphi_{k_0}(x+\tau) - \varphi_{k_0}(x)| + |\varphi_{k_0}(x) - \varphi(x, t_{k_0})| \\ &\quad + |\varphi(x, t_{k_0}) - \varphi(x, t)| \\ &\leq \|u(\cdot, t) - u(\cdot, t_{k_0})\| + \|u(\cdot, t_{k_0}) - u_{k_0}\| \\ &\quad + |\varphi_{k_0}(x+\tau) - \varphi_{k_0}(x)| + \|u(\cdot, t_{k_0}) - u_{k_0}\| \\ &\quad + \|u(\cdot, t_{k_0}) - u(\cdot, t)\| \\ &\leq 2(W + 1)\omega(h) + |\varphi_{k_0}(x+\tau) - \varphi_{k_0}(x)| \\ &< \frac{\epsilon}{2} + |\Phi(x+\tau) - \Phi(x)| < \epsilon. \end{aligned}$$

It follows that $E(\epsilon/2, N, \Phi) \subset E(\epsilon, N, \varphi)$. This implies that φ is almost automorphic function by Proposition 1.1.

As in the proof for Theorem 3.3 in [26], one shows that $\xi \in \mathcal{PAP}_0(\triangle)$. The proof is complete.

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REFERENCES

1. L. Amerio and G. Prouse, *Almost-periodic Functions and Functional Equations*, Van Nostrand, New York, 1971.
2. B. Basit, *Some problems concerning different types of vector valued almost periodic functions*, Dissertationes Math. **338**(1995).
3. B. Basit and A. J. Pryde, *Polynomials and functions with finite spectra on locally compact Abelian groups*, Bull. Austral. Math. Soc. **51**(1995), 33–42.
4. B. Basit, *Harmonic analysis and asymptotic behavior of solutions to the abstract Cauchy problem*, Monash Univ. Analysis Paper **91**(1994).
5. ———, *Spectral characterization of abstract functions*, Analele stiintificale univ. "Ai.I. Cuza" din Iasi **28**(1982), 25–34.
6. ———, *Les fonctions abstraites presque automorphiques et presques almost periodiques au sens de Levitan et leurs differences*, Bull. Sci. Math. (2)**101**(1977), 131–148.
7. ———, *Note on a theorem of Levitan for the integral of almost periodic functions*, Rend. Ist. Mathem. Univ. Trieste **5**(1973), 9–14.
8. G. Baskakov, *Spectral criteria for almost periodicity of solution of functional equations*, Math. Notes of Acad. of Sci. USSR. **24**(1978), 606–612.
9. J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, Wiley, Now York, 1989.
10. S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A. **48**(1962), 2039–2043.
11. C. Corduneanu, *Almost Periodic Functions*, Wiley, New York, 1968.
12. N. Dunford and J. T. Schwartz, *Linear Operators,II*, Interscience, New York, 1963.
13. W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. **67**(1949), 217–240.
14. P. Flor, *Rhytmische abbildung abelescher gruppen II* Z. Wahrscheinlichkeitstheorie verw Geb. **7**(1967), 17–28.
15. S. Goldberg and P. Irwin, *Weakly almost periodic vector valued functions*, Dissertationes Math. **157**(1979).
16. B. M. Levitan, *Integration of almost periodic function with values in Banach spaces*, Izv. Akad. Nauk. **30**(1966), 1101–1110.
17. B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.
18. F. Lust, *Elements ergodiques et totalement ergodiques dans $L^\infty(\Gamma)$*, Studia Math. **69**(1981), 191–225.
19. P. Milnes, *Almost automorphic functions and totally bounded groups*, Rocky Mount. J. Math. **7**(1977), 231–250.
20. V. Q. Phóng, *Almost periodic solutions of Voltera equations*, Differential and Integral Equations **7**(1994), 1083–1093.
21. H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, Oxford, 1968.
22. W. M. Ruess and W. H. Summers, *Ergodic theorems for semigroups of operators*, Proc. Amer. Math. Soc. **114**(1992), 423–432.
23. W. M. Ruess and V. Q. Phóng, *Asymptotically almost periodic solutions of evolution equations in Banach spaces*, J. Differential Equations, to appear.
24. W. A. Veech, *Almost automorphic functions on groups*, Amer. J. Math. **87**(1965), 719–751.

25. S. Zaidman, *Almost Periodic Functions in Abstract Spaces*, Pitman, Boston, London, Melbourne, 1985.
26. C. Zhang, *Pseudo almost periodic solutions of some differential equations*, J. Math. Anal. Appl. **181**(1994), 62–76.
27. ______, *Integration of vector-valued pseudo almost periodic functions*, Proc. Amer. Math. Soc. **121**(1994), 167–174.
28. ______, *Vector-valued pseudo almost periodic functions*, Czech. Math. J., to appear.

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THE $3x+1$ CONJUGACY MAP

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ABSTRACT. The $3x+1$ map T and the shift map S are defined by $T(x) = (3x+1)/2$ for x odd, $T(x) = x/2$ for x even, while $S(x) = (x-1)/2$ for x odd, $S(x) = x/2$ for x even. The $3x+1$ conjugacy map Φ on the 2-adic integers $\mathbf{Z}_2$ conjugates S to T, *i.e.*, $\Phi \circ S \circ \Phi^{-1} = T$. The map Φ mod 2^n induces a permutation Φ_n on $\mathbf{Z}/2^n\mathbf{Z}$. We study the cycle structure of Φ_n. In particular we show that it has order 2^{n-4} for $n \geq 6$. We also count 1-cycles of Φ_n for n up to 1000; the results suggest that Φ has exactly two odd fixed points. The results generalize to the $ax+b$ map, where ab is odd.

1. Introduction.

The $3x+1$ problem concerns iteration of the $3x+1$ function

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod 2 \\ x/2 & \text{if } x \equiv 0 \pmod 2 \end{cases} \tag{1.1}$$

on the integers $\mathbf{Z}$. The well-known $3x+1$ Conjecture asserts that, for each positive integer n, some iterate $T^k(n)$ equals 1, *i.e.*, all orbits on the positive integers eventually reach the cycle $\{1,2\}$.

The $3x+1$ function (1.1) is defined on the larger domain $\mathbf{Z}_2$ of 2-adic integers. It is a measure-preserving map on $\mathbf{Z}_2$ with respect to the 2-adic measure, and it is strongly mixing, so it is ergodic; see [8]. More is true. Let $S\colon \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2$ be the 2-*adic shift map* defined by

$$S(x) = \begin{cases} (x-1)/2 & \text{if } x \equiv 1 \pmod 2 \\ x/2 & \text{if } x \equiv 0 \pmod 2; \end{cases} \tag{1.2}$$

i.e., $S(\sum_{i=0}^{\infty} b_i 2^i) = \sum_{i=0}^{\infty} b_{i+1} 2^i$, if each b_i is 0 or 1. Then T is topologically conjugate to S: there is a homeomorphism $\Phi\colon \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2$ with

$$\Phi \circ S \circ \Phi^{-1} = T. \tag{1.3}$$

In fact T is metrically conjugate to S: one map Φ satisfying (1.3) preserves the 2-adic measure. Thus T is Bernoulli.

The map Φ is determined by (1.3) up to multiplication on the right by an automorphism of the shift S. It is known that the automorphism group of S is isomorphic to $\mathbf{Z}/2\mathbf{Z}$, with nontrivial element $V(x) = -1-x$. (See [6, Theorem 6.9] and the introduction to [3].) We obtain a unique function Φ by adding to (1.3) the side condition $\Phi(0) = 0$. We call Φ the $3x+1$ *conjugacy map*. This function has been constructed several times, apparently first in [8], where Φ^{-1} is denoted Q_∞, and also in [1], [2].

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An important property of Φ is that it is solenoidal. Here we say that a function f on $\mathbf{Z}_2$ is *solenoidal* if, for every n, it induces a function mod 2^n, *i.e.*,

$$x \equiv y \pmod{2^n} \Longrightarrow f(x) \equiv f(y) \pmod{2^n}.$$

This solenoidal property, together with $\Phi(0) = 0$, implies that

$$\Phi(x) \equiv x \pmod{2}. \tag{1.4}$$

For completeness, we give a self-contained proof that Φ is unique. Let Φ and Φ' be two invertible functions satisfying (1.3) and (1.4). Write Q and Q' for their inverses. Then $S \circ Q = Q \circ T$ and $S \circ Q' = Q' \circ T$, and (1.4) gives $Q \equiv Q' \pmod{2}$. If $Q \equiv Q' \pmod{2^k}$ then $Q \circ T = Q' \circ T \pmod{2^k}$, so $S \circ Q \equiv S \circ Q' \pmod{2^k}$. Now $S \circ Q$ and $S \circ Q'$ agree in the bottom k bits, and Q and Q' agree in the bottom bit, so Q and Q' agree in the bottom $k+1$ bits. Hence $Q \equiv Q' \pmod{2^{k+1}}$. By induction $Q \equiv Q' \pmod{2^k}$ for every k, so $Q = Q'$, so $\Phi = \Phi'$.

There is an explicit formula for Φ^{-1} ([8]). Let T^m denote the m-th iterate of T. Then

$$\Phi^{-1}(x) = \sum_{i=0}^{\infty} \left(T^i(x) \bmod 2\right) 2^i. \tag{1.5}$$

This implies (1.3) and (1.4), and also shows that Φ^{-1} is solenoidal.

There is also an explicit formula for Φ ([2]). For $x \in \mathbf{Z}_2$, expand x as

$$x = \sum_l 2^{d_l},$$

in which $\{d_l\}$ is a finite or infinite sequence with $0 \leq d_1 < d_2 < \cdots$. Then

$$\Phi(x) = -\sum_l 3^{-l} 2^{d_l}. \tag{1.6}$$

This also implies (1.3) and (1.4), and shows that Φ is solenoidal.

Various properties of the $3x+1$ map under iteration can be formulated in terms of properties of Φ. The $3x+1$ Conjecture is reformulated as follows ([2], [8]). Here $\mathbf{Z}^+$ denotes the positive integers.

$3x+1$ CONJECTURE. $\mathbf{Z}^+ \subseteq \Phi(\frac{1}{3}\mathbf{Z})$.

Furthermore, it is known that $\Phi(\mathbf{Q} \cap \mathbf{Z}_2) \subseteq \mathbf{Q} \cap \mathbf{Z}_2$. (This is easily proven from (1.6); see [2].) The following conjecture is proposed in [8].

PERIODICITY CONJECTURE. $\Phi(\mathbf{Q} \cap \mathbf{Z}_2) = \mathbf{Q} \cap \mathbf{Z}_2$.

This would imply that the $3x+1$ function T has no divergent trajectories on $\mathbf{Z}$. Recall that a trajectory $\{T^k(n) : k \geq 1\}$ is *divergent* if it contains an infinite number of distinct elements, so that $|T^k(n)| \to \infty$ as $k \to \infty$. In fact, if

$$T_{3,k}(x) = \begin{cases} (3x+k)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}, \end{cases}$$

then the Periodicity Conjecture is equivalent to the assertion that, for all $k \equiv \pm 1 \pmod 6$, the $3x + k$ function has no divergent trajectories on $\mathbf{Z}$. (This follows from [9, Corollary 2.1b].)

This paper studies the $3x + 1$ conjugacy map Φ for its own sake. The function Φ is a solenoidal bijection; it induces permutations Φ_n of $\mathbf{Z}/2^n\mathbf{Z}$. Our object is to determine properties of the cycle structure of the permutations Φ_n. In effect, our results give information about the iterates Φ^k of Φ. We prove in particular that Φ_n contains three "long" cycles of length 2^{n-4}, for all $n \geq 6$.

We remark that the results we prove are not related to the $3x + 1$ Conjecture in any immediate way; indeed for the iterates T^k the conjugacy (1.3) gives $\Phi \circ S^k \circ \Phi^{-1} = T^k$, a relation which does not involve Φ^k for any $k \geq 2$. We do note that the Periodicity Conjecture is equivalent, for any $k \geq 1$, to the assertion that $\Phi^k(\mathbf{Q} \cap \mathbf{Z}_2) = \mathbf{Q} \cap \mathbf{Z}_2$. Consequently information about Φ^k may conceivably prove useful in resolving the Periodicity Conjecture.

The contents of the paper are as follows. In Section 2 we give a table of the cycle lengths of Φ_n for $n \leq 20$. This table motivated our results. We also give data on 1-cycles of Φ_n for $n \leq 1000$. We conjecture that Φ has exactly two odd fixed points. In Section 3 we formulate results on the progressive stabilization of the "long" cycles of Φ_n. In Section 4 we generalize these results to the conjugacy map for the $ax + b$ function

$$T_{a,b}(x) = \begin{cases} (ax+b)/2 & \text{if } x \equiv 1 \pmod 2 \\ x/2 & \text{if } x \equiv 0 \pmod 2), \end{cases}$$

where ab is odd. We prove all these results in Section 5. The proofs are based on Theorem 5.1, which keeps track of the highest-order significant bit in the orbit of $x \bmod 2^{n+2}$. In Section 6 we reconsider "short" cycles of Φ_n, and present a heuristic argument that relates their asymptotics to the number of global periodic points. This heuristic is consistent with the data on 1-cycles presented in Section 2.

There are two appendices on solenoidal maps. Appendix A shows the equivalence of "solenoidal bijection," "solenoidal homeomorphism," and "2-adic isometry." Appendix B shows that a wide class of functions U generalizing the $3x+1$ map T are conjugate to the 2-adic shift S by a solenoidal conjugacy map Φ_U.

Finally, we note that, for odd k, the map $Q(x) = kx$ conjugates the $3x+1$ function to the $3x+k$ function; *i.e.*, $Q \circ T \circ Q^{-1} = T_{3,k}$. Thus the cycle structure of the permutations mod 2^n of all the conjugacy maps $\Phi_{3,k}$ are identical. Other properties of the $3x + 1$ conjugacy map appear in [2], [10], [11]. In particular, Φ and Φ^{-1} are nowhere differentiable on $\mathbf{Z}_2$; see [10], [2].

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2. **Empirical Data and Two Conjectures.**

By (1.4), Φ_n takes odd numbers to odd numbers. Let $\hat{\Phi}_n \colon (\mathbf{Z}/2^n\mathbf{Z})^* \longrightarrow (\mathbf{Z}/2^n\mathbf{Z})^*$ denote its restriction. The properties of Φ_n are completely determined by $\hat{\Phi}_n$. Indeed, $\Phi(2^j x) = 2^j\Phi(x)$ by (1.6), so the action of $\hat{\Phi}_{n-j}$ describes the action of Φ_n on odd numbers times 2^j.

n	$\hat{\Phi}_n$	order($\hat{\Phi}_n$)
2	identity	1
3	$\{1,5\}$	2
4	$\{1,5\}\{9,13\}$	2
5	$\{1,21\}\{5,17\}\{7,23\}\{9,29,25,13\}$	4
6	$\{1,21\}\{3,35\}\{5,17,37,49\}\{7,23\}\{9,29,25,13\}$ $\{19,51\}\{27,59\}\{33,53\}\{39,55\}\{41,61,57,45\}$	4

Table 2.1. Cycle structure of $\hat{\Phi}_n$, $n \leq 6$. 1-cycles are omitted.

Each $\hat{\Phi}_n$ consists of cycles of various lengths, all of which are powers of 2. (See Section 3 for a proof.) The exact form of $\hat{\Phi}_n$ for $n \leq 6$ appears in Table 2.1.

Table 2.2 below lists the number of cycles of various lengths in $\hat{\Phi}_n$ for $n \leq 20$. Let $X_{n,j}$ denote the set of cycles of $\hat{\Phi}_n$ of period 2^j, and let $|X_{n,j}|$ be the number of such cycles. From Table 2.2 we see, empirically, that

$$\text{order}(\hat{\Phi}_n) = 2^{n-4}, \quad n \geq 6. \tag{2.1}$$

We also see a progressive stabilization of the number of "long" cycles in $\hat{\Phi}_n$. In Sections 3–5 we prove both these facts.

How does $|X_{n,j}|$, the number of cycles of $\hat{\Phi}_n$ of size 2^j, behave as $n \longrightarrow \infty$, for fixed j? We give data for the simplest case $|X_{n,0}|$ of 1-cycles. Table 2.3 gives all values of $|X_{n,0}|$ for $n \leq 100$, and Table 2.4 gives values of $|X_{n,0}|$ at intervals of 10 for $n \leq 1000$. We computed the values $|X_{n,0}|$ recursively for increasing n by tracking each 1-cycle individually.

The tables show that $|X_{n,0}|$ behaves irregularly, but has a general tendency to increase. In Section 6 we present a heuristic model which suggests that

$$|X_{n,0}| \sim F_0 n \quad \text{as } n \longrightarrow \infty, \tag{2.2}$$

where F_0 is the number of odd fixed points of Φ. Comparison with Table 2.4 suggests the following conjecture.

FIXED POINT CONJECTURE. *The* $3x + 1$ *conjugacy map* Φ *has exactly two odd fixed points.*

We searched for odd rational fixed points, and immediately found two: $x = -1$ and $x = 1/3$. The conjecture thus asserts that these are the only odd fixed points of Φ. We do not know of any approach to determine the existence or non-existence of non-rational odd fixed points.

More generally we propose the following conjecture.

(n,j)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2																
3	2	1															
4	4	2															
5	6	3	1														
6	6	7	3														
7	8	10	3	3													
8	14	17	8	0	3												
9	14	21	18	4	0	3											
10	10	35	24	14	2	0	3										
11	12	40	37	18	12	2	0	3									
12	16	48	70	23	16	10	2	0	3								
13	26	53	79	60	24	11	10	2	0	3							
14	22	63	111	98	50	14	11	10	2	0	3						
15	18	81	129	153	84	40	11	11	10	2	0	3					
16	20	96	179	186	137	78	31	11	11	10	2	0	3				
17	18	91	242	236	207	131	61	29	11	11	10	2	0	3			
18	12	104	305	308	312	192	105	56	29	11	11	10	2	0	3		
19	16	86	375	401	432	307	152	99	54	29	11	11	10	2	0	3	
20	26	95	424	573	564	445	281	133	91	54	29	11	11	10	2	0	3

Table 2.2. Number of cycles $|X_{n,j}|$ of $\hat{\Phi}_n$ of order 2^j, $0 \leq j \leq n$.

$3x+1$ CONJUGACY FINITENESS CONJECTURE. *For each $j \geq 0$, the $3x+1$ conjugacy map Φ has finitely many odd periodic points of period 2^j.*

We have no idea whether the $3x+1$ conjugacy map Φ has finitely many odd periodic points in total. There are examples of $ax+b$ conjugacy maps that have no odd periodic points; see Section 4.

3. Cycle structure of Φ_n: Inert Cycles and Stable Cycles.

There is a simple relation between the cycles of Φ_n and those of Φ_{n+1}: For $x \in \mathbf{Z}_2$, the cycle $\sigma_{n+1}(x)$ that x belongs to in Φ_{n+1} has length $|\sigma_{n+1}(x)|$ either equal to or double the length of the cycle $\sigma_n(x)$ that x belongs to in Φ_n.

This follows from a more general fact. Call a function $f_{n+1}: \mathbf{Z}/m^{n+1}\mathbf{Z} \longrightarrow \mathbf{Z}/m^{n+1}\mathbf{Z}$ *consistent mod m^n* if it induces a function f_n from $\mathbf{Z}/m^n\mathbf{Z}$ to $\mathbf{Z}/m^n\mathbf{Z}$, *i.e.*, if

$$x_1 \equiv x_2 \pmod{m^n} \Longrightarrow f_{n+1}(x_1) \equiv f_{n+1}(x_2) \pmod{m^n}. \tag{3.1}$$

LEMMA 3.1. *Let $f_{n+1}: \mathbf{Z}/m^{n+1}\mathbf{Z} \longrightarrow \mathbf{Z}/m^{n+1}\mathbf{Z}$ be a function which is consistent mod m^n. If x is a purely periodic point of f_{n+1} then x is a purely periodic point of f_n and*

$$|\sigma_{n+1}(x)| = k\,|\sigma_n(x)|$$

for some integer k with $1 \leq k \leq m$.

(k,j)	0	1	2	3	4	5	6	7	8	9
1		12	32	52	80	116	106	152	124	110
2	2	16	38	54	82	122	112	144	124	108
3	2	26	36	56	96	124	110	120	130	108
4	4	22	38	54	106	124	112	108	128	92
5	6	18	36	54	116	114	106	114	128	96
6	6	20	36	54	90	128	92	132	136	96
7	8	18	50	68	82	118	106	140	124	102
8	14	12	60	68	92	94	116	144	118	108
9	14	16	62	84	102	92	122	144	104	88
10	10	26	50	92	108	100	132	144	98	90

Table 2.3. Number of 1-cycles in $\hat{\Phi}_{10j+k}$.

(k,j)	0	1	2	3	4	5	6	7	8	9
1	10	96	380	700	844	1278	1078	1330	1944	2030
2	26	90	458	788	840	1176	1130	1142	2180	2162
3	50	116	452	916	1134	1000	1212	1170	2194	2230
4	92	156	544	780	942	914	1270	1240	2226	2128
5	108	240	574	678	874	998	1462	1346	2130	2206
6	100	278	588	908	910	1110	1476	1538	2294	2362
7	132	282	628	818	866	1172	1360	1562	2204	2354
8	144	320	634	784	932	1172	1358	1778	2184	2362
9	98	378	784	870	1060	1072	1190	1974	2114	2242
10	90	404	714	892	1150	1086	1208	1808	2056	2308

Table 2.4. Number of 1-cycles in $\hat{\Phi}_{100j+10k}$.

PROOF. The image of $\sigma_{n+1}(x)$ under projection mod m^n consists of k copies of a purely periodic orbit $\sigma_n(x)$, for some $k \geq 1$. The bound $k \leq m$ follows because any element of $\mathbf{Z}/m^n\mathbf{Z}$ has only m distinct preimages in $\mathbf{Z}/m^{n+1}\mathbf{Z}$. ∎

Lemma 3.1 applies to Φ_{n+1}, because Φ is solenoidal. Since $m = 2$ we have

$$|\sigma_{n+1}(x)| = k\,|\sigma_n(x)| \text{ with } k = 1 \text{ or } 2.$$

We call a cycle $\sigma_{n+1}(x)$ *split* if $|\sigma_{n+1}(x)| = |\sigma_n(x)|$, because $\sigma_n(x)$ lifts to two cycles mod 2^{n+1}, namely $\sigma_{n+1}(x)$ and $\sigma_{n+1}(x) + 2^n$. If $|\sigma_{n+1}(x)| = 2\,|\sigma_n(x)|$ we call $\sigma_{n+1}(x)$ *inert*, because $\sigma_n(x)$ has lifted to a single cycle. If $\sigma_{n+1}(x)$ is an inert cycle, and $|\sigma_n(x)| = p$, then $|\sigma_{n+1}(x)| = 2p$ and

$$\Phi^p_{n+1}(x) \equiv x + 2^n \pmod{2^{n+1}}. \tag{3.2}$$

By induction on n, the length of any cycle $|\sigma_n(x)|$ is a power of 2.

We call a cycle $\sigma_n(x)$ *stable* if $\sigma_m(x)$ is an inert cycle for all $m \geq n$. If $\sigma_n(x)$ is a stable cycle, then

$$|\sigma_m(x)| = 2^{m-n+1}\,|\sigma_{n-1}(x)|, \quad m \geq n.$$

For a stable cycle $\sigma_n(x)$, Lemma 3.1 guarantees that the map Φ restricted to

$$\{y \in \mathbf{Z}_2 : y \equiv x_i \pmod{2^n} \text{ for some } x_i \in \sigma_n(x)\}$$

has no periodic points.

Our main result concerning Φ is as follows.

THEOREM 3.1. *For the* $3x+1$ *conjugacy map* Φ, *suppose that* $|\sigma_n(x)| \geq 4$ *and that* $\sigma_n(x)$ *and* $\sigma_{n+1}(x)$ *are both inert cycles. Then* $\sigma_{n+2}(x)$ *is also an inert cycle. Consequently* $\sigma_n(x)$ *is a stable cycle.*

Theorem 3.1 follows from Corollary 5.1 at the end of Section 5.

The hypothesis $|\sigma_n(x)| \geq 4$ is necessary in Theorem 3.1. For example, $\sigma_5(3) = \{3\}$, so both $\sigma_6(3) = \{3, 35\}$ and $\sigma_7(3) = \{3, 99, 67, 35\}$ are inert, but $\sigma_8(3) = \{3, 227, 195, 163\}$ is split.

COROLLARY 3.1A. $\operatorname{order}(\hat{\Phi}_n) = \operatorname{order}(\Phi_n) = 2^{n-4}$, *for* $n \geq 6$.

PROOF. $\sigma_6(5) = \{5, 17, 37, 49\}$ is stable. ∎

We next consider Table 2.2 in light of Theorem 3.1. Again let $X_{n,j}$ denote the set of cycles of $\hat{\Phi}_n$ of period 2^j. Call $X_{n,j}$ *stabilized* if it consists entirely of stable cycles.

COROLLARY 3.1B. *Assume that all* $X_{n,n-j}$ *are stabilized for* $0 \leq j \leq k-1$, *and that* $|X_{n,n-k}| = |X_{n+1,n+1-k}| = |X_{n+2,n+2-k}|$. *Then* $X_{m,m-k}$ *is stabilized for* $m \geq n$, *and* $|X_{m,m-k}| = |X_{n,n-k}|$.

This criterion gives the stabilized region indicated in Table 2.2. For $n = 20$ over 90% of all elements in $(\mathbf{Z}/2^n\mathbf{Z})^*$ are in stable cycles.

4. The $ax+b$ Conjugacy Map.

Consider now the $ax+b$ function

$$T_{a,b}(x) = \begin{cases} (ax+b)/2 & \text{if } x \equiv 1 \pmod 2 \\ x/2 & \text{if } x \equiv 0 \pmod 2, \end{cases} \tag{4.1}$$

where ab is odd. See [4], [5], [7], and [12] for various properties of $T_{a,b}$ under iteration on $\mathbf{Z}$.

The 2-adic shift map S is conjugate to the general $ax+b$ function $T_{a,b}$ by the $ax+b$ *conjugacy map* $\Phi_{a,b}\colon \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2$; *i.e.*, $\Phi_{a,b} \circ S \circ \Phi_{a,b}^{-1} = T_{a,b}$. If $x = \sum_l 2^{d_l}$, where $\{d_l\}$ is a finite or infinite sequence with $0 \leq d_1 < d_2 < \cdots$, then

$$\Phi_{a,b}(x) = -b \sum_l a^{-l} 2^{d_l}; \tag{4.2}$$

see [2]. Associated to $\Phi_{a,b}$ are the permutations $\Phi_{a,b,n}$ on $\mathbf{Z}/2^n\mathbf{Z}$ obtained by reducing $\Phi_{a,b}$ mod 2^n. The following result generalizes Theorem 3.1.

THEOREM 4.1. *For the $ax+b$ conjugacy map $\Phi_{a,b}$, suppose that a cycle $\sigma_n(x)$ of $\Phi_{a,b,n}$ has $|\sigma_n(x)| \geq 4$.*

(i) If $a \equiv 1 \pmod 4$, and $\sigma_n(x)$ is an inert cycle, then $\sigma_{n+1}(x)$ is an inert cycle.

(ii) If $a \equiv 3 \pmod 4$, and $\sigma_n(x)$ and $\sigma_{n+1}(x)$ are both inert cycles, then $\sigma_{n+2}(x)$ is an inert cycle.

This theorem follows from Corollary 5.1 in Section 5. The proof actually shows that in case (i) the weaker hypothesis $|\sigma_n(x)| \geq 2$ suffices, when $b \equiv 3 \pmod 4$.

There are examples of $ax+b$ conjugacy maps $\Phi_{a,b}$ for which all cycles eventually become stable. Such $\Phi_{a,b}$ then have no odd periodic points. Using Theorem 4.1 we easily check that the $25x-3$ conjugacy map when taken mod 32 has an odd part consisting of two stable cycles of period 8.

5. **The Highest Order Bit.**

Throughout this section, $\Phi = \Phi_{a,b}$ is a general $ax+b$ conjugacy map, where a and b are odd. We analyze the high bit of the iterates of $\Phi \bmod 2^{n+2}$. All earlier results follow from Theorem 5.1 below.

For $x \in \mathbf{Z}_2$, expand x as

$$x = \sum_{k=0}^{\infty} \mathrm{bit}_k(x) 2^k, \tag{5.1}$$

where $\mathrm{bit}_k(x)$ is either 0 or 1. Define the bit sums

$$\mathrm{pop}_k(x) := \sum_{j=0}^{k} \mathrm{bit}_j(x). \tag{5.2}$$

The $ax+b$ conjugacy map is then given by

$$\Phi_{a,b}(x) = \sum_{k=0}^{\infty} \frac{-b}{a^{\mathrm{pop}_k(x)}} \mathrm{bit}_k(x) 2^k, \tag{5.3}$$

by (4.2).

LEMMA 5.1. *If $y, z \in \mathbf{Z}_2$ with $z \equiv y \pmod{2^n}$, then*

$$\begin{aligned} \Phi(z) - \Phi(y) - (z-y) \equiv 2^{n+1}\Big(\frac{ab+1}{2} + \frac{b(a-1)}{2}\mathrm{pop}_{n-1}(y)\Big) \\ \cdot \big(\mathrm{bit}_n(y) + \mathrm{bit}_n(z)\big) \pmod{2^{n+2}}. \end{aligned} \tag{5.4}$$

PROOF. Expand $\Phi(z) - \Phi(y) \pmod{2^{n+2}}$ using (5.3). We have $\mathrm{bit}_k(z) = \mathrm{bit}_k(y)$ and $\mathrm{pop}_k(z) = \mathrm{pop}_k(y)$ for $0 \leq k \leq n-1$, so the first n terms in $\Phi(z) - \Phi(y)$ cancel. Thus

$$\begin{aligned} \Phi(z) - \Phi(y) \equiv 2^n\Big(\Big(\frac{-b}{a^{\mathrm{pop}_n(z)}}\Big)\mathrm{bit}_n(z) - \Big(\frac{-b}{a^{\mathrm{pop}_n(y)}}\Big)\mathrm{bit}_n(y)\Big) \\ + 2^{n+1}\Big(\Big(\frac{-b}{a^{\mathrm{pop}_{n+1}(z)}}\Big)\mathrm{bit}_{n+1}(z) - \Big(\frac{-b}{a^{\mathrm{pop}_{n+1}(y)}}\Big)\mathrm{bit}_{n+1}(y)\Big). \end{aligned}$$

Substitute $a^{-1} \equiv a \pmod 4$ in the coefficient of 2^n, and $b \equiv a^{-1} \equiv 1 \pmod 2$ in the coefficient of 2^{n+1}:

$$\begin{aligned}\Phi(z) - \Phi(y) \equiv 2^n\big(ba^{\mathrm{pop}_n(y)}\,\mathrm{bit}_n(y) - ba^{\mathrm{pop}_n(z)}\,\mathrm{bit}_n(z)\big)\\ + 2^{n+1}\big(\mathrm{bit}_{n+1}(z) - \mathrm{bit}_{n+1}(y)\big) \pmod{2^{n+2}}.\end{aligned} \tag{5.5}$$

On the other hand

$$z - y \equiv 2^n\big(\mathrm{bit}_n(z) - \mathrm{bit}_n(y)\big) + 2^{n+1}\big(\mathrm{bit}_{n+1}(z) - \mathrm{bit}_{n+1}(y)\big) \pmod{2^{n+2}}. \tag{5.6}$$

Subtract (5.6) from (5.5):

$$\Phi(z) - \Phi(y) - (z-y) \equiv 2^n\big((ba^{\mathrm{pop}_n(y)} + 1)\,\mathrm{bit}_n(y) - (ba^{\mathrm{pop}_n(z)} + 1)\,\mathrm{bit}_n(z)\big) \pmod{2^{n+2}}.$$

Substitute $a^k \equiv 1 + (a-1)k \pmod 4$, $\mathrm{pop}_k(x)\,\mathrm{bit}_k(x) = \big(1 + \mathrm{pop}_{k-1}(x)\big)\,\mathrm{bit}_k(x)$, and then $\mathrm{pop}_{n-1}(z) = \mathrm{pop}_{n-1}(y)$:

$$\begin{aligned}
&\Phi(z) - \Phi(y) - (z-y)\\
&\quad\equiv 2^n\Big(\Big(b\big(1 + (a-1)\,\mathrm{pop}_n(y)\big) + 1\Big)\,\mathrm{bit}_n(y)\\
&\qquad\quad - \Big(b\big(1 + (a-1)\,\mathrm{pop}_n(z)\big) + 1\Big)\,\mathrm{bit}_n(z)\Big)\\
&\quad\equiv 2^n\Big(\big(ab + 1 + b(a-1)\,\mathrm{pop}_{n-1}(y)\big)\,\mathrm{bit}_n(y)\\
&\qquad\quad - \big(ab + 1 + b(a-1)\,\mathrm{pop}_{n-1}(z)\big)\,\mathrm{bit}_n(z)\Big)\\
&\quad\equiv 2^n\big(ab + 1 + b(a-1)\,\mathrm{pop}_{n-1}(y)\big)\big(\mathrm{bit}_n(y) - \mathrm{bit}_n(z)\big)\\
&\quad\equiv 2^{n+1}\Big(\frac{ab+1}{2} + \frac{b(a-1)}{2}\,\mathrm{pop}_{n-1}(y)\Big)\big(\mathrm{bit}_n(y) - \mathrm{bit}_n(z)\big) \pmod{2^{n+2}}.
\end{aligned}$$

This is equivalent to (5.4). ∎

Now fix $x \in \mathbf{Z}_2$, and fix $n \geq 0$. Set $|\sigma_n(x)| = 2^j$ and assume from now on that

$$\sigma_{n+1}(x) \text{ is inert,} \tag{5.7}$$

so that $|\sigma_{n+1}(x)| = 2^{j+1}$. We wish to determine whether or not $\sigma_{n+2}(x)$ is inert. According to (3.2) this occurs if and only if

$$\Phi^{2^{j+1}}(x) \equiv x + 2^{n+1} \pmod{2^{n+2}}. \tag{5.8}$$

We now introduce the quantities

$$e_k[i] := \mathrm{bit}_k\big(\Phi^i(x)\big).$$

In terms of the $e_k[i]$, we have

$$\sigma_{n+2}(x) \text{ is inert} \iff e_{n+1}[0] \neq e_{n+1}[2^{j+1}], \tag{5.9}$$

by (5.8). We proceed to evaluate $e_{n+1}[2^{j+1}] - e_{n+1}[0] \bmod 2$. The main theorems of this paper are deduced from the following formula.

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THEOREM 5.1. *If* $|\sigma_n(x)| = 2^j$ *and* $\sigma_{n+1}(x)$ *is an inert cycle, then*

$$e_{n+1}[2^{j+1}] - e_{n+1}[0] \equiv 1 + \frac{ab+1}{2}2^j + \frac{b(a-1)}{2}N \pmod 2, \tag{5.10}$$

where

$$N = \sum_{i=0}^{2^j-1} \operatorname{pop}_{n-1}\big(\Phi^i(x)\big). \tag{5.11}$$

PROOF. First we define $X_i = \big(\Phi^{i+1+2^j}(x) - \Phi^{i+1}(x)\big) - \big(\Phi^{i+2^j}(x) - \Phi^i(x)\big)$. Since $\sigma_{n+1}(x)$ is an inert cycle, $\Phi^{i+2^j}(x) \equiv \Phi^i(x) + 2^n \pmod{2^{n+1}}$, so, by Lemma 5.1,

$$X_i \equiv 2^{n+1}\left(\frac{ab+1}{2} + \frac{b(a-1)}{2}\operatorname{pop}_{n-1}\big(\Phi^i(x)\big)\right) \pmod{2^{n+2}}.$$

Adding up the X_i gives

$$\sum_{i=0}^{2^j-1} X_i \equiv 2^{n+1}\left(\frac{ab+1}{2}2^j + \frac{b(a-1)}{2}N\right) \pmod{2^{n+2}}. \tag{5.12}$$

Next define $Y_i = 2^n\big((e_n[i+1+2^j] - e_n[i+1]) - (e_n[i+2^j] - e_n[i])\big)$. The sum of the Y_i telescopes:

$$\sum_{i=0}^{2^j-1} Y_i = 2^n(e_n[2^{j+1}] - e_n[2^j] - e_n[2^j] + e_n[0]).$$

Since $\sigma_{n+1}(x)$ is an inert cycle, $e_n[0] = e_n[2^{j+1}] \neq e_n[2^j]$, so

$$\sum_{i=0}^{2^j-1} Y_i = 2^n(2e_n[0] - 2e_n[2^j]) \equiv 2^{n+1} \pmod{2^{n+2}}. \tag{5.13}$$

On the other hand,

$$\begin{aligned} X_i - Y_i &\equiv 2^{n+1}(e_{n+1}[i+1+2^j] - e_{n+1}[i+1] - e_{n+1}[i+2^j] + e_{n+1}[i]) \\ &\equiv 2^{n+1}(e_{n+1}[i+1+2^j] + e_{n+1}[i+1] - e_{n+1}[i+2^j] - e_{n+1}[i]). \end{aligned}$$

In this form the sum of $X_i - Y_i$ also telescopes:

$$\sum_{i=0}^{2^j-1}(X_i - Y_i) \equiv 2^{n+1}(e_{n+1}[2^{j+1}] - e_{n+1}[0]) \pmod{2^{n+2}}.$$

Comparing this sum with (5.12) and (5.13), we get

$$2^{n+1}(e_{n+1}[2^{j+1}] - e_{n+1}[0]) \equiv 2^{n+1}\left(\frac{ab+1}{2}2^j + \frac{b(a-1)}{2}N\right) - 2^{n+1} \pmod{2^{n+2}},$$

which implies (5.10). ∎

COROLLARY 5.1. *(i) If* $a \equiv 1 \pmod 4$, *then*

$$e_{n+1}[2^{j+1}] - e_{n+1}[0] \equiv \begin{cases} 1 \pmod 2 \text{ if } b \equiv 3 \pmod 4 \text{ or } j \geq 1 \\ 0 \pmod 2 \text{ otherwise.} \end{cases}$$

(ii) If $a \equiv 3 \pmod 4$, *and* $\sigma_n(x)$ *is inert, then*

$$e_{n+1}[2^{j+1}] - e_{n+1}[0] \equiv \begin{cases} 1 \pmod 2 \text{ if } j \geq 2, \\ 0 \pmod 2 \text{ if } j = 1. \end{cases}$$

Note that (i) proves Theorem 4.1(i), and (ii) proves Theorem 4.1(ii), using (5.9). Theorem 3.1 then follows as a special case of Theorem 4.1(ii).

PROOF. (i) Here $a \equiv 1 \pmod 4$, so the term involving N in (5.10) drops out.
(ii) Here $a \equiv 3 \pmod 4$, and $j \geq 1$, so (5.10) simplifies to

$$e_{n+1}[2^{j+1}] - e_{n+1}[0] \equiv 1 + N \pmod 2.$$

The inertness of $\sigma_n(x)$ gives

$$\text{bit}_{n-1}\left(\Phi^{i+2^{j-1}}(x)\right) = 1 - \text{bit}_{n-1}\left(\Phi^i(x)\right),$$

so

$$\text{pop}_{n-1}\left(\Phi^{i+2^{j-1}}(x)\right) + \text{pop}_{n-1}\left(\Phi^i(x)\right) \equiv 1 \pmod 2.$$

Thus

$$N = \sum_{i=0}^{2^{j-1}-1} \left(\text{pop}_{n-1}\left(\Phi^{i+2^{j-1}}(x)\right) + \text{pop}_{n-1}\left(\Phi^i(x)\right)\right) \equiv \sum_{i=0}^{2^{j-1}-1} 1 = 2^{j-1} \pmod 2.$$

Now (ii) follows. ∎

6. **Cycle Structure of $\hat{\Phi}_n$: Short Cycles.** We consider the behavior of "short" cycles of the $3x + 1$ conjugacy map; *i.e.*, the behavior of $|X_{n,j}|$ as $n \longrightarrow \infty$ for fixed j. We describe a heuristic model which relates the asymptotics of $|X_{n,j}|$ to the number of global odd periodic points of Φ.

We first note that the odd periodic points $\text{Per}^*(\Phi)$ of Φ determine the entire set $\text{Per}(\Phi)$ of periodic points of Φ. The relation

$$\Phi(2x) = 2\Phi(x) \tag{6.1}$$

implies that x has period 2^j if and only if $2x$ has period 2^j. Thus

$$\text{Per}(\Phi) = \{2^k x : k \geq 0 \text{ and } x \in \text{Per}^*(\Phi)\}. \tag{6.2}$$

Let F_j be the number of orbits of Φ containing an odd periodic point of minimal period 2^j. The $3x + 1$ Conjugacy Finiteness Conjecture of Section 2 asserts that all F_j are finite.

We obtain a simple heuristic model for the 1-cycles $X_{n,1}$ of $\hat{\Phi}_n$ by classifying them into two types: those arising by reduction mod 2^n from an odd fixed point of Φ, and all the rest. Call these "immortal" and "mortal" 1-cycles, respectively. Our heuristic model is to assume that each "mortal" 1-cycle has equal probability of splitting or remaining inert, independently of all other 1-cycles. When a "mortal" 1-cycle splits, both its progeny in $X_{n+1,1}$ are "mortal." An "immortal" 1-cycle in $X_{n,1}$ always splits, and gives rise to two 1-cycles in $X_{n+1,1}$, at least one of which is "immortal." We also assume that only F_0 "immortal" 1-cycles appear in total, *i.e.*, for all large enough n each "immortal" 1-cycle splits into one "immortal" 1-cycle and one "mortal" 1-cycle.

This model is a branching process model with two types of individuals. The expected number of individuals $Z_{n,1}$ at step n is

$$E[Z_{n,1}] = F_0 n + c_0, \tag{6.3}$$

where c_0 is a constant depending on the levels of the initial occurrences of the F_0 "immortal" 1-cycles. The empirical data in Tables 6.3 and 6.4 seem consistent with this model, with $F_0 = 2$. We know that $F_0 \geq 2$ in any case. The two "immortal" 1-cycles that we know of both appear at $n = 1$, so that if $F_0 = 2$, then $c_0 = 0$ in (6.3).

To obtain a heuristic model for $|X_{n,j}|$ when $j \geq 1$, we use a refined classification of cycles of $\hat{\Phi}_n$. A *step* consists of passing from $\hat{\Phi}_{n-1}$ to $\hat{\Phi}_n$. For $0 \leq d \leq j \leq n$ let $X_{n,j,d}$ denote the set of cycles of $\hat{\Phi}_n$ of size 2^j which have remained inert for exactly d steps. Let $Y_{n,j,d}$ denote the subset of $X_{n,j,d}$ that consists of cycles that split in going to $\hat{\Phi}_{n+1}$. Then we have

$$|X_{n+1,j,0}| = 2\sum_{d=0}^{n} |Y_{n,j,d}|$$

and

$$|X_{n+1,j+1,d+1}| = |X_{n,j,d}| - |Y_{n,j,d}|\,.$$

We know the following facts about these quantities:

(1) If a cycle of length at least 8 has been inert for $d \geq 2$ steps, it remains inert. Thus $|Y_{n,j,d}| = 0$ if $j \geq 3$ and $d \geq 2$.

(2) Any cycle of length 4 which has been inert for $d = 2$ steps must split; *i.e.*, $|X_{n,2,2}| = |Y_{n,2,2}|$.

(3) Any odd periodic point x of Φ of period 2^j gives rise to a cycle of period 2^j of $\hat{\Phi}_n$ for all sufficiently large n. This cycle always splits. Such cycles are in both $X_{n,j,0}$ and $Y_{n,j,0}$.

The quantity we are interested in is

$$|X_{n,j}| = \sum_{d=0}^{n} |X_{n,j,d}|\,.$$

The facts above imply that $|X_{n,j}|$ is entirely determined by knowledge of $|X_{m,j,0}|$, $|Y_{m,j,0}|$, and $|Y_{m,j,1}|$, for all $m \leq n$.

Our heuristic model is then to suppose the following:

(1) Each cycle in $X_{n,j,1}$ has (independently) probability $1/2$ of falling in $Y_{n,j,1}$.
(2) Each "mortal" cycle in $X_{n,j,0}$ has (independently) probability $1/2$ of falling in $Y_{n,j,0}$, and if so its two progeny in $X_{n+1,j,0}$ are "mortal."
(3) Each "immortal" cycle in $X_{n,j,0}$ lies in $Y_{n,j,0}$, and one of its progeny in $X_{n+1,j,0}$ is "immortal" and the other is "mortal," with finitely many exceptions.

This is a multi-type branching process model. If $Z_{n,j}$ denotes the total number of individuals in such a process, then one may calculate that, for large n,

$$E[Z_{n,1}] = \frac{1}{4}F_0 n^2 + \left(F_1 + \frac{1}{4}F_0\right)n - F_1 + \frac{1}{2}F_0 + c_1, \tag{6.4}$$

in which c_1 is a constant depending on the initial occurrence of "immortal" cycles. (We assume that $c_0 = 0$.) For $j \geq 2$, where stable cycles may occur, the formula for $E[Z_{n,j}]$ becomes quite complicated.

It might be interesting to further compare predictions of this model for $j \geq 1$ with actual data for Φ. We know of one odd periodic cycle of Φ of length 2, namely $\{1, -1/3\}$; *i.e.*, $\Phi(1) = -1/3$ and $\Phi(-1/3) = 1$. Thus $F_1 \geq 1$.

7. Appendix A. Solenoidal Maps.

Call a map $F: \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2$ *solenoidal* if, for all n,

$$x \equiv y \pmod{2^n} \Longrightarrow F(x) \equiv F(y) \pmod{2^n}. \tag{A.1}$$

An equivalent condition in terms of the 2-adic metric $|\cdot|_2$ is that F is *nonexpanding*; *i.e.*,

$$|F(x) - F(y)|_2 \leq |x - y|_2, \quad \text{all } x, y \in \mathbf{Z}_2. \tag{A.2}$$

If F_1 and F_2 are solenoidal maps, then so is $F_1 \circ F_2$.

Call a family of functions $F_n: \mathbf{Z}/2^n\mathbf{Z} \longrightarrow \mathbf{Z}/2^n\mathbf{Z}$ *compatible* if F_n agrees with F_{n-1} under projection $\pi_n: \mathbf{Z}/2^n\mathbf{Z} \longrightarrow \mathbf{Z}/2^{n-1}\mathbf{Z}$; *i.e.*, if $\pi_n \circ F_n = F_{n-1} \circ \pi_n$. A compatible family $\{F_n\}$ has an *inverse limit* $F: \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2$ defined by

$$F(x) \equiv F_n(x) \pmod{2^n}, \quad \text{for all } n. \tag{A.3}$$

The term "solenoidal" is justified by the following lemma.

LEMMA A.1. *F is solenoidal if and only if F is the inverse limit of a compatible family $\{F_n\}$.*

PROOF. If F is solenoidal, then F mod 2^n induces a function $F_n: \mathbf{Z}/2^n\mathbf{Z} \longrightarrow \mathbf{Z}/2^n\mathbf{Z}$, for each n; and $\{F_n\}$ is a compatible family. The reverse implication follows from (A.3). ∎

LEMMA A.2. *Let U be the inverse limit of a compatible family $\{U_n\}$. Then the following are equivalent.*
(i) U is a bijection.
(ii) For each n, U_n is a permutation.
(iii) For each n, if $U(x) \equiv U(y) \pmod{2^n}$ then $x \equiv y \pmod{2^n}$.

PROOF. (i) ⇒ (ii). U is surjective, so U_n is surjective.

(ii) $\Rightarrow$ (i). Write $V_n = U_n^{-1}$. Then $\{V_n\}$ is a compatible family. Let V be its inverse limit. By construction $U \circ V$ is the inverse limit of identity functions, so $U \circ V$ is the identity. Similarly $V \circ U$ is the identity. Hence U is a bijection.

(ii) $\Rightarrow$ (iii). If $U(x) \equiv U(y) \pmod{2^n}$ then $U_n(x \bmod 2^n) = U_n(y \bmod 2^n)$ so $x \bmod 2^n = y \bmod 2^n$.

(iii) $\Rightarrow$ (ii). Suppose that $U_n(a) = U_n(b)$. Select x and y in $\mathbf{Z}_2$ such that $a = x \bmod 2^n$, $b = y \bmod 2^n$. Then $U_n(x \bmod 2^n) = U_n(y \bmod 2^n)$, so $U(x) \equiv U(y) \pmod{2^n}$, so $x \equiv y \pmod{2^n}$, so $a = b$. ∎

COROLLARY A.3. *The following are equivalent.*

(i) U is a solenoidal bijection.

(ii) U is a solenoidal homeomorphism.

(iii) U is a 2-adic isometry.

U is a *2-adic isometry* if $|U(x) - U(y)|_2 = |x - y|_2$.

PROOF. (i) $\Rightarrow$ (iii). U is solenoidal so $|U(x) - U(y)|_2 \le |x - y|_2$. On the other hand, by Lemma A.1, U is an inverse limit; and U is a bijection, so $|U(x) - U(y)|_2 \ge |x - y|_2$ by Lemma A.2 (i $\Rightarrow$ iii).

(iii) $\Rightarrow$ (ii). Since $|U(x) - U(y)|_2 \le |x - y|_2$, U is solenoidal. By Lemma A.1, U is an inverse limit; by Lemma A.2 (iii $\Rightarrow$ i), U is a bijection. Since $|U(x) - U(y)|_2 \ge |x - y|_2$, U^{-1} is solenoidal. Finally, solenoidal implies continuous.

(ii) $\Rightarrow$ (i). Immediate. ∎

8. Appendix B. Functions Solenoidally Conjugate to the Shift.

For any two solenoidal bijections V_0, V_1 define $U_{V_0,V_1} : \mathbf{Z}_2 \longrightarrow \mathbf{Z}_2$ by

$$U(x) = \begin{cases} V_0(x/2) & \text{if } x \equiv 0 \pmod 2, \\ V_1\big((x-1)/2\big) & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

For example, take $V_0(x) = x$ and $V_1(x) = ax + (a+b)/2$; then U_{V_0,V_1} is the $ax+b$ function.

In this appendix we show that a map is solenoidally conjugate to the 2-adic shift map S—*i.e.*, conjugate to S by a solenoidal bijection—if and only if it is of the form U_{V_0,V_1}.

LEMMA B.1. *Let V be a solenoidal bijection. If $z \equiv w \pmod{2^{m-1}}$ then $V(z) \equiv V(w) + z - w \pmod{2^m}$.*

PROOF. If $z \equiv w \pmod{2^m}$ then $V(z) \equiv V(w) \pmod{2^m}$.

If $z \equiv w + 2^{m-1} \pmod{2^m}$ then still $V(z) \equiv V(w) \pmod{2^{m-1}}$. By Corollary A.3, V is an isometry, so if $V(z) \equiv V(w) \pmod{2^m}$ then $z \equiv w \pmod{2^m}$, contradiction. Thus $V(z) \equiv V(w) + 2^{m-1} \pmod{2^m}$. ∎

LEMMA B.2. *Set $U = U_{V_0,V_1}$. Fix $m \ge 1$. If $y \equiv x + 2^m e \pmod{2^{m+1}}$ then $U(y) \equiv U(x) + 2^{m-1} e \pmod{2^m}$.*

PROOF. Put $b = x \bmod 2$; then $U(x) = V_b\big(S(x)\big)$. Also $U(y) = V_b\big(S(y)\big)$, since $y \equiv x \pmod 2$. We have $S(y) \equiv S(x) + 2^{m-1} e \pmod{2^m}$; by Lemma B.1, $V_b\big(S(y)\big) \equiv V_b\big(S(x)\big) + 2^{m-1} e \pmod{2^m}$. ∎

LEMMA B.3. *Set $U = U_{V_0,V_1}$. Fix $m \ge j \ge 1$. If $y \equiv x + 2^m e \pmod{2^{m+1}}$ then $U^j(y) \equiv U^j(x) + 2^{m-j}e \pmod{2^{m-j+1}}$.*

PROOF. Lemma B.2 and induction on j. ∎

LEMMA B.4. *Set $U = U_{V_0,V_1}$. Fix $m \ge 1$. If $y \equiv x + 2^m e \pmod{2^{m+1}}$ then $U^m(y) \equiv U^m(x) + e \pmod 2$.*

PROOF. Lemma B.3 with $j = m$. ∎

LEMMA B.5. *Set $U = U_{V_0,V_1}$. Fix $b_0, b_1, b_2, \ldots \in \{0,1\}$. Define $x_0 = 0$ and $x_{m+1} = x_m + 2^m\big(b_m - U^m(x_m)\big)$. Then $y \equiv x_m \pmod{2^m}$ if and only if $U^i(y) \equiv b_i \pmod 2$ for $0 \le i < m$.*

PROOF. We induct on m. For $m = 0$ there is nothing to prove.

Say $y \equiv x_{m+1} \pmod{2^{m+1}}$. Then $y \equiv x_m + 2^m\big(b_m - U^m(x_m)\big) \pmod{2^{m+1}}$; by Lemma B.4, $U^m(y) \equiv U^m(x_m) + b_m - U^m(x_m) = b_m \pmod 2$. Also $y \equiv x_m \pmod{2^m}$, so by the inductive hypothesis $U^i(y) \equiv b_i \pmod 2$ for $0 \le i < m$.

Conversely, say $U^i(y) \equiv b_i \pmod 2$ for $0 \le i \le m$. By the inductive hypothesis $y \equiv x_m \pmod{2^m}$. Write $y = x_m + 2^m e$. Then $b_m \equiv U^m(y) \equiv U^m(x_m) + e \pmod 2$ by Lemma B.4. Thus $y \equiv x_m + 2^m\big(b_m - U^m(x_m)\big) = x_{m+1} \pmod{2^{m+1}}$. ∎

THEOREM B.1. *Set $U = U_{V_0,V_1}$. Define $Q(x) = \sum_{m=0}^{\infty}\big(U^m(x) \bmod 2\big)2^m$. Then Q is a solenoidal bijection, and $U = Q^{-1} \circ S \circ Q$.*

Thus any map of the form U_{V_0,V_1} is solenoidally conjugate to S. (See Theorem B.2 below for the converse.) Q^{-1} generalizes the $ax + b$ conjugacy map.

PROOF. Injective: Say $Q(y) = Q(x)$. Define $b_m = U^m(x) \bmod 2$; then $U^m(y) \equiv U^m(x) \equiv b_m \pmod 2$. Next define $x_0 = 0$ and $x_{m+1} = x_m + 2^m\big(b_m - U^m(x_m)\big)$. By Lemma B.5, $y \equiv x_m \pmod{2^m}$ and $x \equiv x_m \pmod{2^m}$. Thus $y \equiv x \pmod{2^m}$ for every m; i.e., $y = x$.

Solenoidal: Say $y \equiv x \pmod{2^n}$. Define $b_m = U^m(x) \bmod 2$, $x_0 = 0$, and $x_{m+1} = x_m + 2^m\big(b_m - U^m(x_m)\big)$. Then $x \equiv x_n \pmod{2^n}$ by Lemma B.5, so $y \equiv x_n \pmod{2^n}$; by Lemma B.5 again, $U^m(y) \equiv b_m \pmod 2$ for $0 \le m < n$. Thus $Q(y) \equiv Q(x) \pmod{2^n}$.

Surjective: Given $b = \sum_{i=0}^{\infty} b_i 2^i$ with $b_i \in \{0,1\}$, define $x_0 = 0$ and $x_{m+1} = x_m + 2^m\big(b_m - U^m(x_m)\big)$. Since $x_{m+1} \equiv x_m \pmod{2^m}$ the sequence $x_1, x_2, \ldots$ converges to a 2-adic limit y, with $y \equiv x_m \pmod{2^m}$. By Lemma B.5, $U^m(y) \equiv b_m \pmod 2$ for all m. Thus $Q(y) = b$.

Finally, it is immediate from the definition of Q that $Q \circ U = S \circ Q$. ∎

THEOREM B.2. *Let Q be a solenoidal bijection. Define $U = Q^{-1} \circ S \circ Q$. Then $U = U_{V_0,V_1}$ for some solenoidal bijections V_0, V_1.*

PROOF. If $Q(0)$ is even then $Q^{-1}(x) \equiv x \pmod 2$ for all x; so write

$$Q^{-1}(x) = \begin{cases} 2W_0(x/2) & \text{if } x \equiv 0 \pmod 2, \\ 1 + 2W_1\big((x-1)/2\big) & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

Then W_0, W_1 are solenoidal bijections, and $U = U_{V_0,V_1}$ where $V_i = Q \circ W_i$.

Similarly, if $Q(0)$ is odd then $Q^{-1}(x) \equiv -1 - x \pmod 2$ for all x; so write

$$Q^{-1}(x) = \begin{cases} 1 + 2W_0(x/2) & \text{if } x \equiv 0 \pmod 2, \\ 2W_1\big((x-1)/2\big) & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

Again W_0, W_1 are solenoidal bijections, and $U = U_{V_0,V_1}$ where $V_i = Q \circ W_i$. ∎

REFERENCES

1. E. Akin, *Why is the* 3x + 1 *problem so hard?* Math. Magazine, to appear.
2. D. J. Bernstein, *A non-iterative* 2*-adic statement of the* 3x+1 *conjecture*, Proc. Amer. Math. Soc. **121**(1994), 405–408.
3. M. Boyle, J. Franks and B. Kitchens, *Automorphisms of one-sided subshifts of finite type*, Ergod. Th. Dyn. Sys. **10**(1990), 421–449.
4. R. E. Crandall, *On the '*3x + 1*' problem*, Math. Comp. **32**(1978), 1281–1292.
5. Z. Franco and C. Pomerance, *On a conjecture of Crandall concerning the QX* + 1 *problem*, Math. Comp. **49**(1995), to appear.
6. G. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory **3**(1969), 320–375.
7. E. Heppner, *Eine Bemerkung zum Hasse-Syracuse Algorithmus*, Arch. Math. **31**(1978), 317–320.
8. J. C. Lagarias, *The* 3x + 1 *problem and its generalizations*, Amer. Math. Monthly **92**(1985), 3–23.
9. ———, *The set of rational cycles for the* 3x + 1 *problem*, Acta Arithmetica **56**(1990), 33–53.
10. H. Müller, *Das '*3n + 1*' Problem*, Mitteilungen der Math. Ges. Hamburg **12**(1991), 231–251.
11. ———, *Über eine Klasse* 2*-adischer Funktionen im Zussamenhang mit dem "*3x+1*"-Problem*, Abh. Math. Sem. Univ. Hamburg **64**(1994), 293–302.
12. R. P. Steiner, *On the "QX* + 1 *problem," Q odd, I, II*, Fibonacci Quart. **19**(1981), 285–288, 293–296.

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FINITE SUBGROUPS IN INTEGRAL GROUP RINGS

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ABSTRACT. A p-subgroup version of the conjecture of Zassenhaus is proved for some finite solvable groups including solvable groups in which any Sylow p-subgroup is either abelian or generalized quaternion, solvable Frobenius groups, nilpotent-by-nilpotent groups and solvable groups whose orders are not divisible by the fourth power of any prime.

1. **Introduction** Let $\mathcal{U}_1\mathbb{Z}G$ denote the group of units of augmentation one of the integral group ring of a finite group G. The Zassenhaus conjecture (ZC3) says that any finite subgroup of $\mathcal{U}_1\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to a subgroup of G (see [19, Chapter 5]). Its particular case (ZC1) states that the same is true for torsion units of $\mathcal{U}_1\mathbb{Z}G$. We know that (ZC3) holds for nilpotent groups [22] and for split metacyclic groups ([15], [21]). K. W. Roggenkamp and L. Scott have shown that the Zassenhaus conjecture is false and a counterexample is a finite metabelian group [11]. However, somewhat weaker statements hold for large families of finite and infinite groups (see [19, Chapters 5 and 6] and [1], [3], [4], [7], [8], [9], [10], [12]). In the present paper we consider the following p-subgroup version of (ZC3).

$(p-ZC3)$ If H is a p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ then there exists a unit $\alpha \in \mathbb{Q}G$ such that $\alpha^{-1}H\alpha \subset G$.

In particular, if (p-ZC3) is true for a group G then any Sylow p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ is rationally conjugate to a p-subgroup of G. Conjugation of those Sylow subgroups of $\mathcal{U}_1\mathbb{Z}G$ which can be embedded into a group basis is investigated in [9], [10].

In this paper all groups G are assumed to be finite. In Section 2 we establish a reduction modulo a normal subgroup. We apply it to generalize a result of [16] and to prove (p-ZC3) for nilpotent-by-nilpotent groups. In particular, this conjecture is true for both metabelian and supersolvable groups. We also give a partial solution of Problem 32 of [19] and point out that (p-ZC3) implies a positive solution of that problem. In Sections 3 and 4 we establish (ZC3) for S_4 and a covering group of it, the Binary Octahedral Group. We apply these results in Section 5 to prove (p-ZC3) for solvable groups in which any Sylow subgroup is either abelian or generalized quaternion. As a consequence we deduce (p-ZC3) for solvable Frobenius groups. We also prove (p-ZC3) for a family of groups including those solvable groups whose orders are not divisible by the fourth power of any prime.

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2. A reduction step and some applications. For an element $\alpha = \sum_{g \in G} \alpha(g) g$ of $\mathbb{Z}G$ we put $\tilde{\alpha}(g) = \sum_{h \in C_g} \alpha(h)$ where C_g is the conjugacy class of $g \in G$. We use symbol $\sim$ to denote conjugation in a group.

Let N be a normal subgroup of G, $\bar{G} = G/N$, $\Psi\colon \mathbb{Z}G \longrightarrow \mathbb{Z}(G/N)$ the natural map, $\bar{g} = \Psi(g)$ for $g \in G$. This notation will be used in all what follows.

LEMMA 2.1. *Let $\alpha \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit, $\beta = \Psi(\alpha)$ and $\big(o(\alpha), |N|\big) = 1$. If the order of $g \in G$ is relatively prime to $|N|$ then $\tilde{\alpha}(g) = \tilde{\beta}(\bar{g})$.*

PROOF. Set $S_g = \{h \in G : \bar{h} \sim \bar{g} \text{ in } \bar{G}\}$ and $S'_g = \{h \in S_g : o(h) = o(g)\}$. We see that $\tilde{\beta}(\bar{g}) = \sum_{h \in S_g} \alpha(h)$. Note that if h is not in S'_g then $\big(o(h), |N|\big) \neq 1$ and consequently there is a prime p such that $p \mid o(h)$ but p does not divide $o(\alpha)$. By [19, Lemma 38.11], $\tilde{\alpha}(h) = 0$. Since the complement of S'_g in S_g is a normal subset of G, we have that $\tilde{\beta}(\bar{g}) = \sum_{h \in S'_g} \alpha(h)$. It suffices to show that the elements of S'_g are conjugate to g. Indeed, if $h \in S'_g$ then $t^{-1}ht = g\theta$ for some $t \in G$, $\theta \in N$ and the equality $o(h) = o(g)$ implies that the cyclic subgroups $\langle g \rangle$ and $\langle g\theta \rangle$ are complements for N in $N \rtimes \langle g \rangle$. Since $\big(o(g), |N|\big) = 1$, we get, by Schur-Zassenhaus Theorem, that $g\theta$ is conjugate to g. The result follows. ∎

The next result generalizes [7, Lemma 2.3].

THEOREM 2.2. *Let H be a finite subgroup of $\mathcal{U}_1\mathbb{Z}G$ such that $(|H|, |N|) = 1$ and G_0 be a subgroup of G with $(|G_0|, |N|) = 1$. Then H is rationally conjugate to G_0 iff $\Psi(H)$ is conjugate to $\Psi(G_0)$ in $\mathbb{Q}\bar{G}$.*

PROOF. We only have to prove the converse. Let $\bar{H} = \Psi(H)$ and $\bar{G}_0 = \Psi(G_0)$. Let $\gamma^{-1}\bar{H}\gamma = \overline{G_0}$ for some $\gamma \in \mathbb{Q}\bar{G}$, $\alpha \in H$ and β be as above. We see that $h_\alpha = \gamma^{-1}\beta\gamma$ is, up to conjugacy, the unique element of $\bar{G}$ with $\tilde{\beta}(h_\alpha) \neq 0$. From [19, Lemma 38.11] it follows that $\big(o(h_\alpha), |N|\big) = 1$. From the Schur-Zassenhaus Theorem it follows that we can choose $g_\alpha \in G$ such that $h_\alpha = \Psi(g_\alpha)$ and $\big(o(g_\alpha), |N|\big) = 1$. Then it follows from [19, Lemma 38.11] and Lemma 2.1 that, up to conjugacy, g_α is the unique element of G with $\tilde{\alpha}(g_\alpha) \neq 0$. Since $(|G_0|, |N|) = 1$, the restriction of Ψ to G_0 gives an isomorphism between G_0 and $\bar{G}_0$. Denote by Ψ_1 the inverse of this isomorphism and define a homomorphism $\phi\colon H \longrightarrow G_0$ by setting $\phi(\alpha) = \Psi_1(\gamma^{-1}\beta\gamma)$. Since $\Big(o\big(\phi(\alpha)\big), |N|\Big) = 1$, Lemma 2.1 implies that $\tilde{\alpha}\big(\phi(\alpha)\big) = \tilde{\beta}\big(\Psi\phi(\alpha)\big) = \tilde{\beta}(h_\alpha) \neq 0$ and $\phi(\alpha)$ is conjugate to g_α. It follows by [19, Lemma 41.4] that H is rationally conjugate to G_0. ∎

REMARK. We have proved that if $H < \mathcal{U}_1\mathbb{Z}G$ and $(|H|, |N|) = 1$ then Ψ is injective on H.

As a consequence we have the following:

COROLLARY 2.3. *Suppose that (ZC3) holds for the factor group G/N. Then any finite subgroup $H \subset \mathcal{U}_1\mathbb{Z}G$ whose order is relatively prime to the order of N is rationally conjugate to a subgroup of G.*

We also obtain some consequences for split extensions.

COROLLARY 2.4. *Let G be an extension of a nilpotent group N by a group X which satisfies (p-ZC3). If the orders of N and X are relatively prime then G satisfies (p-ZC3).*

PROOF. Let H be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. If p does not divide the order of N then we use Theorem 2.2 and the assumption on X. If p does divide $|N|$ then G has a normal Sylow p-subgroup and hence, by [19, Theorem 41.12], we obtain that H is rationally conjugate to a subgroup of G. ∎

We give now an improvement of Lemma 37.13 of [19].

LEMMA 2.5. *Let $G = N \rtimes X$, where the orders of N and X are relatively prime, and let $\alpha = vw \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit with $v \in \mathcal{U}\big(1 + \Delta(G,N)\big)$ and $w \in \mathcal{U}_1\mathbb{Z}X$. If $\big(o(\alpha), |N|\big) = 1$ then α and w are rationally conjugate.*

PROOF. We observe that $\tilde{\alpha}(g) = \tilde{w}(g)$ for all $g \in G$. Indeed, if $\big(o(g), |N|\big) \neq 1$ then it follows from [19, Lemma 38.11] that $\tilde{\alpha}(g) = \tilde{w}(g) = 0$. If $\big(o(g), |N|\big) = 1$ then, by Schur-Zassenhaus Theorem, we may suppose that $g \in X$ and apply Lemma 2.1.

Now let d be a divisor of $o(\alpha)$. Then $\alpha^d = v_d w^d$ with $v_d \in \mathcal{U}\big(1 + \Delta(G,N)\big)$ and we use the same reasoning for the units α^d, w^d. Hence, according to [13, Theorem 2], α and w are conjugate in $\mathbb{Q}G$. ∎

The next result is a modification of Lemma 37.6 of [19].

LEMMA 2.6. *Let H_1 and H_2 be isomorphic finite subgroups of $\mathcal{U}_1\mathbb{Z}G$ with a given isomorphism $\varphi: H_1 \longrightarrow H_2$. Suppose that $\chi(h) = \chi\big(\varphi(h)\big)$ for all $h \in H_1$ and all absolutely irreducible characters χ of G. Then H_1 is conjugate to H_2 in $\mathbb{Q}G$.*

PROOF. We extend the representation $\Gamma: G \longrightarrow \mathrm{Gl}(n, \mathbb{C})$ corresponding to χ linearly to $\Gamma_1: H_1 \longrightarrow \mathrm{Gl}(n, \mathbb{C})$. By assumption the characters of Γ_1 and $\Gamma_1\varphi$ are equal and, consequently, the images of H_1 and H_2 are conjugate in any simple component of $\mathbb{C}G$. Hence H_1 is conjugate to H_2 in $\mathbb{C}G$ and Lemma 37.5 of [19] implies that the conjugation can be taken in $\mathbb{Q}G$. ∎

Now we extend Theorem 37.17 of [19].

THEOREM 2.7. *Let G be as in Lemma 2.5. Then any finite subgroup H of $\mathcal{U}_1\mathbb{Z}G$ with $(|H|, |N|) = 1$ is rationally conjugate to a subgroup of $\mathcal{U}_1\mathbb{Z}X$.*

PROOF. For $\alpha \in H$ we write $\alpha = vw$ with $v \in \mathcal{U}\big(1 + \Delta(G,N)\big)$ and $w \in \mathcal{U}_1\mathbb{Z}X$. By Lemma 2.5 the isomorphism $H \ni \alpha = vw \longrightarrow w$ satisfies the hypothesis of Lemma 2.6. Hence H is conjugate to H_0 in $\mathbb{Q}G$, where H_0 is the image of H in $\mathcal{U}_1\mathbb{Z}X$. ∎

COROLLARY 2.8. *Let N be a normal subgroup of G and H be a finite subgroup in $1 + \Delta(G,N)$. If p is a prime which divides $|H|$ then p divides $|N|$. In particular, if N is a Hall subgroup of G then $|H|$ divides $|N|$.*

PROOF. We already know that $|H|$ is a divisor of $|G|$. Suppose that there exists a prime p that divides the order of H and does not divide $|N|$. Let $\alpha \in H$ be a unit of order p. Then $\Psi(\alpha) = 1$ and, by Theorem 2.2, we have that α is rationally conjugate to 1, a contradiction. ∎

THEOREM 2.9. *Let G be a nilpotent-by-nilpotent group. Then (p-ZC3) holds for G.*

PROOF. Let H be a p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ and G_1 be a normal nilpotent subgroup of G so that G/G_1 is nilpotent. If G_1 is not a p-group, then G possesses a normal p'-subgroup N. It follows from Theorem 2.2 and induction on the order of G that H is conjugate in $\mathbb{Q}G$ to a subgroup of G. If G_1 is a p-group, then the Sylow p-subgroup of G is normal and [19, Lemma 41.12] implies that H is rationally conjugate to a subgroup of G. ∎

The proof of the following lemma can be found in [7] (see Lemma 1.5).

LEMMA 2.10. *Let G be a solvable group and P an abelian Sylow p-subgroup of G. If P is not normal in G then $O_{p'}(G) \neq 1$.*

PROPOSITION 2.11. *Let P be an abelian Sylow p-subgroup of a solvable group G. If H is a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$ then H is rationally conjugate to a subgroup of G.*

PROOF. By [19, Theorem 41.12] we may assume that P is not normal in G. It follows from the preceding lemma that $N = O_{p'}(G) \neq 1$. Since the factor group G/N satisfies our hypothesis we can use Theorem 2.2 and induction to conclude that H is rationally conjugate to a subgroup of G. ∎

S. K. Sehgal has proposed the following question [19, Problem 32]:
Let $u \in \mathcal{U}\big(1 + \Delta(G, N)\big)$ be a torsion unit with $N \lhd G$. Does $o(u)$ divide $|N|$?
Now we point out that (p-ZC3) implies a positive solution of this question.

PROPOSITION 2.12. *Let N be a normal subgroup of a group G which satisfies (p-ZC3). If H is a finite subgroup of $\mathcal{U}\big(1 + \Delta(G, N)\big)$ then $|H|$ divides $|N|$.*

PROOF. Let H_p be a Sylow p-subgroup of H. By (p-ZC3), H_p is rationally conjugate to a subgroup H_0 of G. Going down modulo N we see that $H_0 \subset N$. Hence $|H_0|$ divides $|N|$, and consequently $|H|$ divides $|N|$. ∎

Note that Corollary 2.8 gives a partial solution of this problem.

3. (ZC3) for S_4.

The Zassenhaus conjecture for cyclic subgroups in $\mathbb{Z}S_4$ was proved in [5]. In this section we prove the following:

THEOREM 3.1. *(ZC3) holds for S_4.*

PROOF. Let $G = S_4$ and let H be a finite subgroup of $\mathcal{U}_1\mathbb{Z}G$. It is known that G has a faithful irreducible complex representation $\Gamma\colon G \longrightarrow \mathrm{Gl}(3, \mathbb{C})$ such that the trace of $\Gamma\big((12)\big)$ is 1. We denote also by Γ the extension of this representation to $\mathbb{Z}G$. Since (ZC1) holds for G it follows that Γ is faithful on H. Therefore

$$|\Gamma(H)| = |H|. \tag{3.2}$$

Denoting by F the Fitting subgroup of G we have that $F = \langle (12)(34), (13)(24) \rangle$ and $G/F \cong S_3$. Since F is abelian, there exists an invertible matrix X such that $X^{-1}\Gamma(F)X$ has a diagonal form. It is easy to see that

$$X^{-1}\Gamma(F)X = \{I, \mathrm{diag}(-1, -1, 1), \mathrm{diag}(-1, 1, -1), \mathrm{diag}(1, -1, -1)\}. \tag{3.3}$$

Denote by Ψ the natural map $\mathbb{Z}G \longrightarrow \mathbb{Z}G/F$, $\bar{H} = \Psi(H)$ and $H_0 = H \bigcap \big(1 + \triangle(G, F)\big)$. In view of (ZC1), going down modulo F, we obtain that

$$h \in H_0 \text{ if and only if } \gamma^{-1}h\gamma \in F \text{ for some unit } \gamma \in \mathbb{Q}G. \tag{3.4}$$

We may also assume that H is not cyclic. According to Lemma 2.6 it suffices to find a monomorphism $\varphi: H \longrightarrow G$ such that $h \sim \varphi(h)$ in $\mathbb{Q}G$ for all $h \in H$. We consider several cases.

CASE 1. $H = \langle u, v\rangle$ is isomorphic to the Klein four group. Since $|H|$ divides $|G|$, the order of $\bar{H}$ divides 6 and $[H : H_0] = 1$ or 2.

If the index is 1 then, by (3.4), the map $\varphi: H \longrightarrow F$ defined by $\varphi(u) = (12)(34)$, $\varphi(v) = (13)(24)$ is a group isomorphism such that $h \sim \varphi(h)$ in $\mathbb{Q}G$ for all $h \in H$. Thus, H is rationally conjugate to F.

Suppose now that $[H : H_0] = 2$. Choose generators u, v such that $u \notin H_0$ and $H_0 = \langle v\rangle$. We have that $u \sim (12)$ and $v \sim (12)(34)$ in $\mathbb{Q}G$. Clearly $uv \notin H_0$ and, therefore, $uv \sim (12) \sim (34)$ in $\mathbb{Q}G$. We now define an isomorphism $\varphi: H \longrightarrow \langle (12), (12)(34)\rangle$ by putting $\varphi(u) = (12)$, $\varphi(v) = (12)(34)$. Then h is rationally conjugate to $\varphi(h)$ for all $h \in H$ and consequently H is conjugate in $\mathbb{Q}G$ to a subgroup of G.

CASE 2. The order of H is 8. Note that in this case $[H : H_0] = 2$. We show that H is not abelian. First suppose that H is elementary abelian and let u_1, u_2, u_3 be generators of H such that $H_0 = \langle u_2, u_3\rangle$. There exists a matrix Y such that $Y^{-1}\Gamma(H)Y$ consists of diagonal matrices. For $h \in H$ we put $d(h) = Y^{-1}\Gamma(h)Y$. Note that $Y^{-1}H_0Y$ consists of the diagonal matrices given in (3.3). So there is $u \in H_0$ so that $d(u) = \operatorname{diag}(1, -1, -1)$. Now since u_1 does not belong to H_0 we may suppose that $d(u_1) = \operatorname{diag}(-1, 1, 1)$. Hence $d(uu_1) = \operatorname{diag}(-1, -1, -1)$, a contradiction since uu_1 is rationally conjugate to (12).

Let $H = \langle u, v\rangle$, where $o(u^2) = o(v) = 2$. Note that u does not belong to H_0 and, consequently, H_0 is generated by u^2 and v. Let Y be such that $Y^{-1}\Gamma(H)Y$ is in diagonal form. As above, the diagonal form of H_0 consists of the matrices given in (3.3). Since $u^2 \in H_0$ we may assume that $d(u^2) = \operatorname{diag}(-1, -1, 1)$. Hence, $d(u) = \operatorname{diag}(\pm i, \pm i, \pm 1)$. Choose $w \in H_0$ so that $d(w) = \operatorname{diag}(1, -1, -1)$. The element uw has order 4 so, since (ZC1) holds for G, we see that uw is rationally conjugate to (1234). Hence, $d(u)$ and $d(uw)$ are conjugate. However, it is easy to check that the matrices $\operatorname{diag}(\pm i, \pm i, \pm 1)$ and $\operatorname{diag}(\pm i, \pm i, \pm 1)\operatorname{diag}(1, -1, -1)$ are not conjugate in $\mathrm{Gl}(3, \mathbb{C})$, a contradiction.

Thus H is not abelian and since H_0 has exponent 2 we see that H must be isomorphic to the dihedral group of order 8. Let $H = \langle u, v : u^4 = v^2 = 1, v^{-1}uv = u^3\rangle$. Then u is not in H_0 and we may choose v such that H_0 is generated by u^2 and uv. By (3.4), the nontrivial elements of H_0 are conjugate to (12)(34). Since (ZC1) holds for G we have that the other elements of order 2 are rationally conjugate to $(13) \sim (24)$ and those of order 4 are conjugate to (1234). Put $H_1 = \langle (1234), (13)\rangle$ and define an isomorphism of H to H_1 given by $\varphi(u) = (1234)$, $\varphi(v) = (13)$. Then it is clear that h and $\varphi(h)$ are rationally conjugate for all $h \in H$, and hence $H \sim H_1$ in $\mathbb{Q}G$.

CASE 3. The order of H is 6. Since (ZC1) holds for G we must have that H is isomorphic to S_3. Let $H = \langle u, v\rangle$ with $u^3 = v^2 = 1$. Note that H_0 has to be trivial, otherwise H would be cyclic. Hence the elements of order 2 in H are rationally conjugate to (12). Define a monomorphism $\varphi: H \longrightarrow G$ by $\varphi(u) = (123)$ and $\varphi(v) = (12)$. Then it is clear that h and $\varphi(h)$ are rationally conjugate for all $h \in H$ and hence H is conjugate in $\mathbb{Q}G$ to a subgroup of G.

CASE 4. The order of H is 12. Since $\mathcal{U}_1\mathbb{Z}G$ does not have elements of order 6 we have, by [2, pp. 134–135], that H is isomorphic to A_4. Then the elements of order 2 are pairwise conjugate in H and case 1 implies that H_0 is rationally conjugate to F. If $\varphi: H \longrightarrow A_4$ is any isomorphism then, clearly, h is rationally conjugate to $\varphi(h)$ for all $h \in H$. Hence, H is rationally conjugate to A_4.

CASE 5. H is a group basis. We shall show that H is isomorphic to S_4. First note that H is solvable. Put $H_1 = O_{2'}(H)$. Note that H_0 is normal and has order 4 in this case. So if H_1 is not trivial then H would have an element of order 6 which is, obviously, a contradiction. According to case 2, the Sylow 2-subgroups of H are dihedral of order 8. Hence, [6, p. 462] implies that H is isomorphic to S_4. Denote by φ the extension of any isomorphism $G \cong H$ to the integral group rings. It follows from [19, Theorem 43.6] that φ is an inner automorphism induced by a unit of $\mathbb{Q}G$. Consequently, H is rationally conjugate to G. ∎

4. **(ZC3) for the binary octahedral group.** Let G be the Binary Octahedral group. We know that the center $Z(G)$ of G is cyclic of order 2, $G/Z(G) \cong S_4$, the Sylow 2-subgroups of G are generalized quaternion of order 16 and that any group with these properties is isomorphic to G (see, for example, [20, 2.1.14]). Moreover, the Fitting subgroup F of G is isomorphic to the quaternion group of order8 and $G/F \cong S_3$. Let $N = Z(G) = \langle z\rangle$ and let $\Psi: \mathbb{Z}G \longrightarrow \mathbb{Z}G/N$ be the natural map.

LEMMA 4.1. *We can choose a Sylow 2-subgroup* $P = \langle a, b : a^8 = 1, b^2 = a^4, b^{-1}ab = a^{-1}\rangle$ *of G and its generators so that* $\Psi(a) = (1234)$, $F = \langle a^2, ab\rangle$ *and* $a^2 \sim ab$ *in G.*

PROOF. Obviously, we can take a P with $\Psi(P) = \langle(1234), (13)\rangle$. Since F is the inverse image of $\langle(12)(34), (13)(24)\rangle$, we see that $F = \langle a^2, ab\rangle$. Let $x \in G$ be an element of order 3. Then $x^{-1}a^2x \neq a^6$. Going down modulo N to S_4 we see that $x^{-1}a^2x \in \{a^{2k+1}b\}$ and consequently $a^2 \sim ab$ in G. ∎

We also note that a is not conjugate to a^5 in G. For if $x^{-1}ax = a^5$ for some $x \in G$ then $x^{-1}Px = P$ as $\langle a^2, ab\rangle$ is the Fitting subgroup. However, $N_G(P) = P$ and consequently $x \in P$, a contradiction.

In all that follows in this section we choose P and its generators as in the lemma above. If $c \in G$ is an element of order 3, then we obtain, looking at S_4, the following representatives of the conjugacy classes of G:

order of an element	1	2	3	4	6	8
representatives	1	z	c	a^2, b	zc	a, a^5

We note that Ψ maps the two conjugacy classes of elements of order 4 of G to the two classes of elements of order 2 in S_4. We begin by proving that the Zassenhaus conjecture holds for cyclic subgroups in $\mathbb{Z}G$. We say that $\alpha \in \mathbb{Z}G$ satisfies the unique trace property if there exists a $g \in G$, unique up to conjugacy in G, such that $\tilde{\alpha}(g) \neq 0$.

PROPOSITION 4.2. *(ZC1) holds for G.*

PROOF. Let $\alpha \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit, β its image in $\mathbb{Z}S_4$ and $g \in G$. Denote by $\bar{g} = \Psi(g)$. Since (ZC1) holds for S_4 we have that

$$\tilde{\beta}(\bar{g}) \in \{0, 1\}. \tag{4.3}$$

Note first that the unit group $\mathcal{U}_1\mathbb{Z}G$ has a unique element of order 2. So we may suppose that the order of α is not 2. If $o(\alpha) = 3$ then we apply Theorem 2.2. If the order of α is 6 then we may write $\alpha = z\alpha_0$, where the order of α_0 is 3 and so we are done by the previous case. Going down modulo N we see easily that the only possibilities left for the order of α are 4 and 8.

Let α be a 2-element such that $o(\alpha) \geq 4$. We want to show that every element of $\langle\alpha\rangle$ has the unique trace property. Note that z does not belong to the support of α. If g has order 3 or 6 then [19, Lemma 38.11] implies that $\tilde{\alpha}(g) = 0$. So we may suppose that g is of order 4 or 8. Let g and g_0 be elements of G whose orders are 4 and 8 respectively. Going down modulo N it is easy to see that

$$\begin{aligned} \tilde{\beta}(\bar{g}) &= \tilde{\alpha}(g), \\ \tilde{\beta}(\bar{g}_0) &= \tilde{\alpha}(g_0) + \tilde{\alpha}(g_0^5). \end{aligned} \tag{4.4}$$

Since g_0 is not conjugate to g_0^5 in G, there exists an absolutely irreducible character χ of G so that $\chi(g_0) \neq \chi(g_0^5)$. It is easy to see that the degree of χ divides 4 and χ is not zero on an element of order 8. Moreover, χ is faithful as $\Psi(a) = \Psi(a^5)$. Let Γ be the representation associated with χ. Then $\Gamma(z) = -I$ and therefore

$$\chi({g_0}^5) = -\chi(g_0). \tag{4.5}$$

We now treat separately the remaining two cases.

Assume first that α has order 4. It follows from (4.3) and (4.4) that $\tilde{\alpha}(g_0) + \tilde{\alpha}({g_0}^5) = 0$ and that there exists a unique, up to conjugacy, element $g_1 \in G$ of order 4 such that $\tilde{\alpha}(g_1) \neq 0$. Applying χ to α and using (4.3) and (4.5) we obtain that $\chi(\alpha) = \chi(g_1) + 2\tilde{\alpha}(g_0)\chi(g_0)$. It follows from the equalities $g_1^2 = z = \alpha^2$ that the eigenvalues of $\Gamma(\alpha)$ and $\Gamma(g_1)$ are $\pm i$. Note that in G every element is conjugate to its inverse so χ is real-valued. Consequently, $\chi(\alpha) = \chi(g_1) = 0$ and so $\tilde{\alpha}(g_0) = 0$. Thus any element of $\langle\alpha\rangle$ has the unique trace property and in view of [19, Lemma 41.5] $\alpha \sim g_1$ in $\mathbb{Q}G$.

Finally assume that $o(\alpha) = 8$. By the same reasoning we obtain that $\tilde{\alpha}(g) = 0$ if $o(g) \neq 8$ and $\tilde{\alpha}(g_0) + \tilde{\alpha}(g_0^5) = 1$. Hence,

$$\chi(\alpha) = [\tilde{\alpha}(g_0) - \tilde{\alpha}(g_0^5)]\chi(g_0) = [2\tilde{\alpha}(g_0) - 1]\chi(g_0). \tag{4.6}$$

The equalities $\alpha^4 = z = g_0^4$ imply that the eigenvalues of α and g_0 are primitive roots of unity of degree 8. Since χ is real-valued and $\chi(g_0) \neq 0$ we see easily that the only possibilities for $\chi(\alpha)$ and $\chi(g_0)$ are $\pm\sqrt{2}$ and $\pm 2\sqrt{2}$. Using this fact and (4.6) we obtain that $2\tilde{\alpha}(g_0) - 1 = \pm 1$ and so $\tilde{\alpha}(g_0)$ is 0 or 1. It follows from the former case, that every element of $\langle\alpha\rangle$ has the unique trace property and so, by [19, Lemma 41.5], either $\alpha \sim g_0$ or $\alpha \sim g_0^5$ in $\mathbb{Q}G$. ∎

THEOREM 4.7. *G satisfies (ZC3).*

PROOF. As we already mentioned $\mathcal{U}_1\mathbb{Z}G$ has a unique element z of order 2, which is central, and we denoted $N = \langle z\rangle$. So if H is a finite non-cyclic subgroup of $\mathcal{U}_1\mathbb{Z}G$ then the Sylow 2-subgroups of H are either cyclic, or quaternion of order 8 or generalized quaternion of order 16. Moreover, since (ZC3) holds for S_4 and this group does not have subgroups of order 6, $\mathcal{U}_1\mathbb{Z}G$ does not contain subgroups of order 12.

Let $|H| = 8$. Suppose first that $H < \mathcal{U}\big(1 + \Delta(G,F)\big)$. Then, by (ZC1), any $1 \neq h \in H$ is conjugate in $\mathbb{Q}G$ to $a^2 \sim ab$. Therefore, if $\varphi\colon H \longrightarrow F$ is any isomorphism, h is rationally conjugate to $\varphi(h)$ for all $h \in H$, and Lemma 2.6 implies that H and F are conjugate in $\mathbb{Q}G$.

If H is not contained in $\mathcal{U}\big(1 + \Delta(G,F)\big)$ then it is easily seen that, going modulo N, we may choose generators h_0, h_1 of H such that $h_0 \sim b$ and $h_1 \sim a^2$ in $\mathbb{Q}G$. We now define a homomorphism $\varphi\colon H \longrightarrow \langle a^2, b\rangle$ by $\varphi(h_0) = b$, $\varphi(h_1) = a^2$. Since $\Psi(h_1h_0) \notin \mathcal{U}\Big(1 + \Delta\big(S_4, \mathrm{Fit}(S_4)\big)\Big)$ it follows that $h_1h_0 \sim a^2b$ in $\mathbb{Q}G$ and $h_1^3h_0 = zh_1h_0 \sim za^2b = a^6b$ in $\mathbb{Q}G$. Hence h and $\varphi(h)$ are rationally conjugate for all $h \in H$ and consequently so are H and $\langle a^2, b\rangle$.

Suppose now that the order of H is 16. We have that $H \cong P$. Choose generators u, v for H so that $\Psi(u) \sim (1234)$ and $\Psi(v) \sim (13)$ in $\mathbb{Q}S_4$. It follows, by proposition 4.2, that $v \sim b$ in $\mathbb{Q}G$ and either $u \sim a$ or $u \sim a^5$ in $\mathbb{Q}G$. In the later case we consider a^5 instead of a, so we may suppose that $u \sim a$. Define an isomorphism $\varphi\colon H \longrightarrow P$ by $\varphi(u) = a$, $\varphi(v) = b$. Observe that $\Psi(u^kv)$ is rationally conjugate to $(1234)^k(13)$. So if k is even then $\Psi(u^kv) \sim (24) \sim (13)$ and consequently $u^kv \sim b$ in $\mathbb{Q}G$. If k is odd, then $\Psi(u^kv) \sim (14)(23)$ in $\mathbb{Q}S_4$ and, hence $u^kv \sim a^2 \sim ab$ in $\mathbb{Q}G$. So we proved that $h \sim \varphi(h)$ for all $h \in H$ and, therefore, H and P are rationally conjugate.

Let $|H| = 24$. Since S_4 satisfies (ZC3) it follows that $\Psi(H) \sim A_4$ in $\mathbb{Q}S_4$. Since A_4 has a normal Sylow 2-subgroup it follows that H also has a normal Sylow 2-subgroup H_0. Hence $H = H_0 \rtimes \langle v\rangle$ with $v^3 = 1$. Clearly H_0 is the quaternion group of order 8 and as $\Psi(H_0) \sim \Psi(F)$ in $\mathbb{Q}S_4$, going down modulo F, it is easily seen that $H_0 < \mathcal{U}\big(1+\Delta(G,F)\big)$. Consequently, H_0 is rationally conjugate to F. Let $c \in G$ be an element of order 3, $G_1 = F \rtimes \langle c\rangle$ and $\varphi\colon H \longrightarrow G_1$ any isomorphism. Recall that the conjugacy classes of elements of order 3 and 6 are respectively represented by zc and c. From this it easily follows that $\varphi(h) \sim h$ in $\mathbb{Q}G$ for every $h \in H$ and hence H and G_1 are rationally conjugate.

Finally let $|H| = 48$. It follows from the information above that $H/Z(H) \cong S_4$ and the Sylow 2-subgroups of H are isomorphic to P. Hence, H must be the Binary Octahedral

Group. Let $\varphi: H \longrightarrow G$ be any isomorphism. Theorem 3.1 and Proposition 4.2 imply that $\varphi(h) \sim h$ in $\mathbb{Q}G$ for every $h \in H$ with $o(h) \neq 8$. Let $o(h) = 8$ and suppose that $\varphi(h)$ is not rationally conjugate to h. We have that $G = \langle P, c\rangle$, $c^3 = 1$ and $G_1 = F \rtimes \langle c\rangle$ has index 2 in G. Define a map θ by $a \longrightarrow a^5$ and $g \longrightarrow g$ for any $g \in G_1$. Since the elements of G_1 are fixed by this map it follows that it is an automorphism of G. It is easy to check now that if we replace φ by $\varphi\theta$, we get $\varphi(h) \sim h$ in $\mathbb{Q}G$ for all $h \in H$ and consequently H and G are rationally conjugate. ■

5. (p-ZC3) for some solvable groups.

THEOREM 5.1. *Let G be a solvable group such that any Sylow subgroup of G is either abelian or generalized quaternion. Then G satisfies (p-ZC3).*

PROOF. Let H be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. In view of Proposition 2.11 we may assume that $p = 2$ and the Sylow 2-subgroups of G are generalized quaternion. If the Fitting subgroup F of G is not a 2-group, then G contains a non-trivial normal subgroup N of odd order. Since the factor group G/N satisfies the assumption of the theorem we use Theorem 2.2 and induction on $|G|$.

Let F be a 2-group. Since G is solvable, $C_G(F) = Z(F)$ [18, p. 144] and, consequently, $G/Z(F) = N_G(F)/C_G(F)$ is a subgroup of Aut(F). According to [17, Proposition 9.10] if F is not isomorphic to Q_8, the quaternion group of order 8, then Aut(F) is a 2-group and the result follows from [22]. Let $F \cong Q_8$. Then Aut$(F) \cong S_4$, $|Z(F)| = 2$ and, hence, $|G|$ divides 48. By [22] we may suppose that G is not nilpotent. If $|G| = 24$ then G has a normal Sylow 2-subgroup and we can use Theorem 2.9. If $|G| = 48$ then G is the Binary Octahedral group. In this case we apply Theorem 4.7. ■

COROLLARY 5.2. *A finite solvable Frobenius group satisfies (p-ZC3).*

PROOF. By [18, 10.5.6] $G = N \rtimes X$ where N is nilpotent, $(|N|, |X|) = 1$ and the Sylow p-subgroups of X are either abelian or generalized quaternion. Hence, the result follows from Corollary 2.4 and Theorem 5.1. ■

THEOREM 5.3. *Let G be a finite solvable group and L = L(G) the last non-trivial term of the lower central series of G. If p^4 does not divide $|G|$ for any prime p dividing $|L|$, then G satisfies (p-ZC3).*

PROOF. Let H be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. If p does not divide $|L|$ then, since G/L is nilpotent, we apply Theorem 2.2 and the theorem of Weiss [22].

Let p divide $|L|$ and let F be the Fitting subgroup of G. If F is not a p-group, then $N = O_{p'}(F)$ is a non-identity normal subgroup of G. It is easy to see that the factor group G/N satisfies the hypothesis of the theorem, so we may use Theorem 2.2 and induction on the order of G.

Let F be a p-group and P a Sylow p-subgroup of G. In view of Proposition 2.11 and [19, Theorem 41.12] we may assume that P is not abelian and not normal in G. In fact $|P| = p^3$ because p^4 does not divide $|G|$. Now the same arguments as in [7, pp. 4908–4909] shows that $p = 2$ and $G \cong S_4$. Thus, the result follows from Theorem 3.1. ■

REMARK. The proof of the theorem shows that if $H \subset \mathcal{U}_1 \mathbb{Z}G$ is a finite subgroup whose order is relatively prime to that of L then H is rationally conjugate to a subgroup of G.

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REFERENCES

1. A. Bovdi, Z. Marciniak and S. K. Sehgal, *Torsion Units in Infinite Group Rings*, J. Number Theory **47**(1994), 284–299.
2. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Springer-Verlag, Berlin, 1980.
3. M. A. Dokuchaev, *Torsion units in integral group ring of nilpotent metabelian groups*, Comm. Algebra (2) **20**(1992), 423–435.
4. M. A. Dokuchaev and S. K. Sehgal, *Torsion Units in Integral Group Rings of Solvable Groups*, Comm. Algebra (12) **22**(1994), 5005–5020.
5. N. A. Fernandes, *Torsion Units in the Integral Group Ring of S_4*, Bol. Soc. Brasil. Mat. (1) **18**(1987), 1–10.
6. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968.
7. S. O. Juriaans, *Torsion units in integral group rings*, Comm. Algebra (12) **22**(1994), 4905–4913.
8. ———, *Torsion Units in Integral Group Rings II*, Canad. Math. Bull. to appear.
9. W. Kimmerle and K.W. Roggenkamp, *Projective Limits of Group Rings*, J. Pure Appl. Algebra **88**(1993), 119–142.
10. W. Kimmerle, K. W. Roggenkamp and A. Zimmerman, DMV-*Seminar Part I Group Rings: Units and the Isomorphism Problem.*
11. L. Klinger, *Construction of a counterexample to a conjecture of Zassenhaus*, Comm. Algebra **19**(1993), 2303–2330.
12. A. I. Lichtman and S. K. Sehgal, *The elements of finite order in the group of units of group rings of free products of groups*, Comm. Algebra **17**(1989), 2223–2253.
13. I. S. Luthar and I. B. S. Passi, *Zassenhaus conjecture for A_5*, Proc. Indian Acad. Sci. (1) **99**(1989), 1–5.
14. Z. Marciniak, J. Ritter, S. K. Sehgal and A. Weiss, *Torsion units in integral group rings of some metabelian groups, II*, J. Number Theory **25**(1987), 340–352.
15. C. Polcino Milies, J. Ritter and S. K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings II*, Proc. Amer. Math. Soc. (2) **97**(1986), 206–210.
16. C. Polcino Milies and S. K. Sehgal, *Torsion Units in integral group rings of metacyclic groups*, J. Number Theory **19**(1984), 103–114.
17. D. S. Passman, *Permutation groups*, W. A. Benjamin, Inc., New York, 1968.
18. D. J. J. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
19. S. K. Sehgal, *Units of Integral Group Rings*, Longman's, Essex, 1993.
20. M. Shirvani and B. A. F. Wehrfritz, *Skew Linear Groups*, Cambridge, Univ. Press, 1986.
21. A. Valenti, *Torsion Units in Integral Group Rings*, Proc. Amer. Math. Soc. (1) **120**(1994), 1–4.
22. A. Weiss, *Torsion units in integral group rings*, J. Reine Angew. Math. **415**(1991), 175–187.

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MARKOV'S THEOREM FOR ORTHOGONAL MATRIX POLYNOMIALS

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ABSTRACT. Markov's Theorem shows asymptotic behavior of the ratio between the n-th orthonormal polynomial with respect to a positive measure and the n-th polynomial of the second kind. In this paper we extend Markov's Theorem for orthogonal matrix polynomials.

1. **Introduction.** A. Markov established in 1895 (see [M]) the following result, which is now known as Markov's theorem:

$$\lim_{n\to\infty} \frac{q_n(z)}{p_n(z)} = \int_a^b \frac{d\mu(t)}{z-t} \quad \text{for } z \in \mathbb{C} \setminus [a,b],$$

where μ is a probability measure on the finite interval $[a,b]$, $(p_n)_n$ is the sequence of orthonormal polynomials with respect to μ and $(q_n)_n$ is the corresponding sequence of polynomials of the second kind, defined by

$$q_n(x) = \int \frac{p_n(x) - p_n(t)}{x-t}\, d\mu(t), \quad n \geq 0.$$

The hypothesis μ having compact support is too restrictive, and actually the determinacy of the measure μ is a sufficient condition. Even for some families of indeterminate measures Markov's theorem holds (see the recent survey about Markov's theorem [B]).

The purpose of this paper is to extend Markov's theorem for orthogonal matrix polynomials.

We consider a $N \times N$ positive definite matrix of measures W (for any Borel set $A \subset \mathbb{R}$, $W(A)$ is a positive semidefinite numerical matrix), having moments of every order, *i.e.*, the matrix integral

$$\int_{\mathbb{R}} t^n \, dW(t)$$

exists for any nonnegative integer n.

The matrix inner product defined in the usual way by W in the space of matrix polynomials has associated a sequence of orthonormal matrix polynomials $(P_n)_n$, satisfying

$$\int P_n(t)\, dW(t) P_m^*(t) = \delta_{n,m} I, \quad n, m \geq 0.$$

$P_n(t)$ is a matrix polynomial of degree n, with a non-singular leading coefficient and is defined upon a multiplication on the left by a unitary matrix.

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As in the scalar case, the sequence of orthonormal matrix polynomials $(P_n)_n$ satisfies a three-term recurrence relation

$$tP_n(t) = A_{n+1}P_{n+1}(t) + B_nP_n(t) + A_n^*P_{n-1}(t), \quad n \geq 0, \quad P_{-1}(t) = \theta,$$

where A_n are non-singular matrices and B_n are hermitian. Without loss of generality, we assume $P_0(t) = I$. (Here and in the rest of this paper, we write θ for the null matrix, the dimension of which can be determined from the context.)

This three term recurrence relation characterizes the orthonormality of a sequence of matrix polynomials with respect to a positive definite matrix of measures (see, for instance, [AN] or [DL]). In [D1], [D2] and [DV] a very close relationship between orthogonal matrix polynomials and scalar polynomials satisfying a higher order recurrence relation has been established.

The corresponding matrix polynomials of the second kind are defined by

$$Q_n(x) = \int \frac{P_n(x) - P_n(t)}{x - t}\, dW(t).$$

We say that the positive definite matrix of measures W is determinate if no other positive definite matrix of measures has the same moments as those of W, *i.e.*, the positive definite matrix of measures W is uniquely determined by the moments $\int t^n\, dW(t)$ $(n \in \mathbb{N})$.

To establish Markov's theorem, we need the following definitions: Δ_n stands for the set of zeros of the matrix polynomial P_n, *i.e.*, the zeros of $\det(P_n)$. In [DL], it is proved that these zeros are real and have multiplicity at most N. We finally put

$$\Gamma = \cap_{N \geq 0} M_N, \qquad \text{where } M_N = \overline{\cup_{n \geq N} \Delta_n}. \tag{1.1}$$

It is proved in [DL] that orthogonalizing matrix of measures μ for the matrix polynomials $(P_n)_n$ can be found as weak accumulation points of a sequence of discrete measures μ_n with support precisely Δ_n. Therefore, for a determinate matrix of measures W, we have $\operatorname{supp}(W) \subset \Gamma$.

The main result of this paper is the following matrix extension of Markov's Theorem.

THEOREM 1.1. *Assume that W is determinate. Then*

$$\lim_{n \to \infty} P_n^{-1}(z)Q_n(z) = \int \frac{dW(t)}{z - t} \quad \textit{for } z \in \mathbb{C} \setminus \Gamma,$$

and the convergence is uniform for z in compact subsets of $\mathbb{C} \setminus \Gamma$.

We will show that in the matrix version of Markov's theorem, the matrices P_n^{-1} and Q_n must be multiplied in the order $P_n^{-1}Q_n$, otherwise the result could be false.

To prove Markov's theorem we find a quadrature formula (Section 3) for the inner product defined by the matrix of measures W in the space of matrix polynomials. The matrix coefficients, which appear in that quadrature formula, will be given in a closed expression, and this expression will be the key to establish the matrix version of Markov's Theorem (see [SV, and DL] for other versions of the quadrature formula).

In the proof of this quadrature formula we use a property of the zeros (Section 2) of the n-th orthonormal matrix polynomial P_n which will play a fundamental role in the whole theory of orthogonal matrix polynomials:

> If α is a zero of $P_n(t)$ of multiplicity p, then $\operatorname{rank}\big(P_n(\alpha)\big) = N-p$ (hence $p \leq N$), and the matrices $\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(p-1)}(\alpha)$ and $P^*_{n-1}(\alpha)A_n$ define two isomorphisms from the space of right eigenvectors of $P_n(\alpha)$ associated to the eigenvalue 0 into the space of left eigenvectors of $P_n(\alpha)$ associated to the eigenvalue 0. The inverse mapping of $P^*_{n-1}(\alpha)A_n$ is that defined by the matrix $Q_n(\alpha)$.

For a given matrix A, we denote by $\operatorname{Adj}(A)$ the classical adjoint, *i.e.*, the matrix uniquely defined by the property

$$A \operatorname{Adj}(A) = \operatorname{Adj}(A)A = \det(A)I.$$

That the zeros of orthogonal polynomials are simple (in the scalar case) is a well-known property, which is not true in the matrix case. Precisely, the property we have noted above will play in the matrix case the same role as that of the non-multiplicity of the zeros in the scalar case.

To complete this paper, we give a generalization of Markov's theorem showing the asymptotic behavior between the sequence of orthonormal matrix polynomials and the sequence of k-th associated polynomials (Section 5)

2. **Zeros.** We start with the following lemma which contains the matrix version of some classical formulae for orthonormal scalar polynomials. The proofs work as in the scalar case and so are omitted.

LEMMA 2.1. *(1) The Christoffel-Darboux formula and some special cases*

$$\begin{aligned} &P^*_{n-1}(z)A_nP_n(w) - P^*_n(z)A^*_nP_{n-1}(w) \\ &\qquad = (w-z)\sum_{k=0}^{n-1} P^*_k(z)P_k(w), \quad z,w \in \mathbf{C}. \end{aligned} \tag{2.1}$$

$$P^*_{n-1}(z)A_nP_n(z) - P^*_n(z)A^*_nP_{n-1}(z) = \theta, \quad z \in \mathbf{C}. \tag{2.2}$$

$$P^*_{n-1}(z)A_nP'_n(z) - P^*_n(z)A^*_nP'_{n-1}(z) = \sum_{k=0}^{n-1} P^*_k(z)P_k(z), \quad z \in \mathbf{C}. \tag{2.3}$$

(2) Some particular cases of the Green formula:

$$\begin{aligned} &P^*_{n-1}(z)A_nQ_n(w) - P^*_n(z)A^*_nQ_{n-1}(w) \\ &\qquad = I + (w-z)\sum_{k=0}^{n-1} P^*_k(z)Q_k(w), \quad z,w \in \mathbf{C}. \end{aligned} \tag{2.4}$$

$$P^*_{n-1}(z)A_nQ_n(z) - P^*_n(z)A^*_nQ_{n-1}(z) = I, \quad z \in \mathbf{C}. \tag{2.5}$$

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(3) The Liouville-Ostrogadski formula

$$Q_n(z)P_{n-1}^*(z) - P_n(z)Q_{n-1}^*(z) = A_n^{-1}. \tag{2.6}$$

To prove the remarkable property about the zeros of P_n stressed in the introduction, and the quadrature formula, we need the following lemma:

LEMMA 2.2. *Let $A(t)$ be a $N \times N$ matrix polynomial and let a be a zero of $A(t)$ of multiplicity p, i.e., a zero of multiplicity p of the scalar polynomial $\det\big(A(t)\big)$. We put*

$$L(a,A) = \{v \in \mathbb{C}^N : vA(a) = \theta\}, \quad R(a,A) = \{v \in \mathbb{C}^N : A(a)v^* = \theta\}.$$

If $\dim\big(L(a,A)\big) = \dim\big(R(a,A)\big) = p$, then $\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(l)}(a) = \theta$, for $l = 0,\dots,p-2$ and $\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a) \neq \theta$. Moreover $\mathrm{rank}\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a) = p$ and

$$\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)$$

defines a linear mapping from $\mathbb{C}^N$ onto $L(a,A)$ which is an isomorphism from $R(a,A)$ into $L(a,A)$.

PROOF. It follows straightforward from Lemma 2.2 of [DL] that

$$\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(l)}(a) = \theta, \quad \text{for } l = 0,\dots,p-2.$$

We now prove that $\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a) \neq \theta$. Since a is a zero of $A(t)$ of multiplicity p, by differentiating the formula $\mathrm{Adj}\big(A(t)\big)A(t) = \det A(t) I$ and taking into account what we have already proved, we obtain that

$$\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)A(a) = \theta, \quad \text{and} \tag{2.7}$$

$$\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p)}(a)A(a) + \Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)A'(a) = \big(\det A(t)\big)^{(p)}(a)I. \tag{2.8}$$

If $\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a) = \theta$, (2.8) gives that

$$\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p)}(a)A(a) = \big(\det A(t)\big)^{(p)}(a)I.$$

But this is impossible because $A(a)$ is singular but $\big(\det A(t)\big)^{(p)}(a)I$ is non-singular (a is a zero of multiplicity just p of $\det\big(A(t)\big)$).

The formula (2.7) gives that the matrix

$$\Big(\mathrm{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)$$

defines a linear mapping

$$\begin{aligned}\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)\colon \mathbb{C}^N &\longrightarrow L(a,A)\\ v &\longmapsto v\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a).\end{aligned}$$

For $v \in R(a,A)$, $v \neq \theta$, (2.8) gives that

$$v\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)A'(a)v^* \neq \theta.$$

So, the linear mapping defined by $\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)$ is injective in $R(a,A)$. Hence,

$$\operatorname{rank}\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a) \geq \dim\big(R(a,A)\big) = p.$$

But (2.7) gives that the rows of $\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a)$ are vectors of $L(a,A)$, hence

$$\operatorname{rank}\Big(\operatorname{Adj}\big(A(t)\big)\Big)^{(p-1)}(a) \leq \dim\big(L(a,A)\big) = p.$$

And the proof is finished. ∎

Some properties of zeros of orthogonal matrix polynomials on the real line have been established recently in [SV] and [DL] (see also [Z]). We complete here those results by proving the property we noted in the introduction of this paper.

THEOREM 2.3. *If a is a zero of $P_n(t)$ of multiplicity p, then*

(1) a is real, $p \leq N$, $\operatorname{rank}\big(P_n(a)\big) = N - p$, and $\dim\big(R(a,P_n)\big) = \dim\big(L(a,P_n)\big) = p$ ($R(a,P_n)$ and $L(a,P_n)$ are, respectively, the spaces of right and left eigenvectors of $P_n(a)$ associated to the eigenvalue 0).

(2) The matrix $P^*_{n-1}(a)A_n$ defines an isomorphism from $R(a,P_n)$ into $L(a,P_n)$. Its inverse mapping is the isomorphism defined by the matrix $Q_n(a)$.

(3) The matrix $\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(p-1)}(a)$ defines a linear mapping from $\mathbb{C}^N$ onto $L(a,P_n)$ which is an isomorphism from $R(a,P_n)$ into $L(a,P_n)$.

(4) $\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(p-1)}(a)P_n(a) = P_n(a)\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(p-1)}(a) = \theta$ and

$$\operatorname{rank}\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(p-1)}(a) = p,$$

being the p linearly independent rows of $\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(p-1)}(a)$ a basis of the linear space of left eigenvectors of $P_n(a)$ associated to 0.

PROOF. (1) It is contained in parts (1) and (2) of Theorem 1.1 in [DL].

(2) First of all, we prove that if $v \in R(a,P_n)$ then $vP^*_{n-1}(a)A_n \in L(a,P_n)$. Indeed, if $P_n(a)v^* = \theta$, the formula (2.2) gives that $vP^*_{n-1}(a)A_nP_n(a) = \theta$, that is, $vP^*_{n-1}(a)A_n \in L(a,P_n)$.

For a vector $v \in R(a, P_n)$, the formula (2.5) gives

$$vP^*_{n-1}(a)A_nQ_n(a) = v, \tag{2.9}$$

and so, $vP^*_{n-1}(a)A_{n-1} \neq \theta$. This proves that the mapping is injective. Since the dimension of $R(a, P_n)$ and $L(a, P_n)$ is the same, the mapping is automatically an isomorphism.

(2.9) shows that its inverse mapping is just that defined by $Q_n(a)$.

(3) and (4) From (1) we have that $\dim\big(R(a, P_n)\big) = \dim\big(L(a, P_n)\big) = p$. Then (3) and (4) of Theorem 2.3 are straightforwardly deduced from Lemma 2.2. ■

Theorem 2.3 can be extended for perturbations of the orthonormal polynomials. Indeed, let A be a matrix satisfying that A_nA is hermitian, and let us consider the polynomial $P_n(t) - AP_{n-1}(t)$. Then, Theorem 2.3 holds for this polynomial, *i.e.*:

THEOREM 2.4. *If a is a zero of $P_n(t) - AP_{n-1}(t)$ of multiplicity p, where $A_nA = A^*A^*_n$, then*

(1) a is real, $p \leq N$, $\operatorname{rank}\big(P_n(a) - AP_{n-1}(a)\big) = N - p$, and $\dim\big(R(a, P_n - AP_{n-1})\big) = \dim\big(L(a, P_n - AP_{n-1})\big) = p$.

(2) The matrix $P^*_{n-1}(a)A_n$ defines an isomorphism from $R(a, P_n - AP_{n-1})$ into $L(a, P_n - AP_{n-1})$. Its inverse mapping is the isomorphism defined by the matrix $Q_n(a) - AQ_{n-1}(a)$.

(3) The matrix $\Big(\operatorname{Adj}\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(p-1)}(a)$ defines a linear mapping from $\mathbb{C}^N$ onto $L(a, P_n - AP_{n-1})$ which is an isomorphism from $R(a, P_n - AP_{n-1})$ into $L(a, P_n - AP_{n-1})$.

(4)

$$\Big(\operatorname{Adj}\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(p-1)}(a)\big(P_n(a) - AP_{n-1}(a)\big)$$
$$= \big(P_n(a) - AP_{n-1}(a)\big)\Big(\operatorname{Adj}\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(p-1)}(a) = \theta$$

and

$$\operatorname{rank}\Big(\operatorname{Adj}\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(p-1)}(a) = p,$$

being the p linearly independent rows of $\Big(\operatorname{Adj}\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(p-1)}(a)$ a basis of the linear space of left eigenvectors of $P_n(a) - AP_{n-1}(a)$ associated to 0.

By modifying in an appropriate way the formulae showed in Lemma 2.1 (this can be easily done by using that the matrix A_nA is hermitian), the proof of Theorem 2.4 exactly works as that of Theorem 2.3.

3. **The quadrature formula revisited.**

Quadrature formulae have been found recently for orthonormal matrix polynomials by Sinap and Van Assche (see [SV]) and Duran and Lopez-Rodriguez (see [DL]). Here and using other different approach, we improve these quadrature formulae by giving a closed expression for the matrix coefficients which appear in the formula. This expression will be the key to prove the matrix version of Markov's Theorem.

THEOREM 3.1. *Let n be a nonnegative integer. We write $x_{n,k}$ $(k = 1, \dots, m)$ for the different zeros of the matrix polynomial P_n ordered in increasing size (hence $m \le nN$) and $\Gamma_{n,k}$ for the matrices*

$$\Gamma_{n,k} = \frac{1}{\left(\det\left(P_n(t)\right)\right)^{(l_k)}(x_{n,k})}\left(\operatorname{Adj}\left(P_n(t)\right)\right)^{(l_k-1)}(x_{n,k})Q_n(x_{n,k}), \quad k = 1, \dots, m, \tag{3.1}$$

where l_k is the multiplicity of $x_{n,k}$.

(1) For any polynomial P with $\operatorname{dgr}(P) \le 2n-1$ *the following formula holds*

$$\int P(t)\, dW(t) = \sum_{k=1}^{m} P(x_{n,k})\Gamma_{n,k}. \tag{3.2}$$

(2) The matrices $\Gamma_{n,k}$ are positive semidefinite matrices of rank l_k, $k = 1, \dots, m$.

It is worth noting that the smaller the multiplicity of a zero $x_{n,k}$ is, the bigger the singularity of the matrix $\Gamma_{n,k}$ is. For instance, if $x_{n,k}$ is simple, then $\Gamma_{n,k}$ has rank one.

PROOF. To prove (1) of Theorem 3.1 we use the following surprising fact:

Although the zeros of P_n (*i.e.*, the zeros of $\det(P_n)$) can be of multiplicity bigger than one (at most N) the decomposition

$$P(z)P_n^{-1}(z) = \sum_{k=1}^{m} \frac{C_{n,k}}{z - x_{n,k}} \quad \text{if } \operatorname{dgr} P \le n-1, \tag{3.3}$$

is always possible.

Part (4) of Theorem 2.3 and Lemma 2.2 show the reason for this remarkable property: indeed, if a is a zero of multiplicity p for P_n then

$$\operatorname{Adj}\left(P_n(a)\right) = \left(\operatorname{Adj}\left(P_n(t)\right)\right)'(a) = \cdots = \left(\operatorname{Adj}\left(P_n(t)\right)\right)^{(p-2)}(a) = \theta, \quad \text{and}$$
$$\left(\operatorname{Adj}\left(P_n(t)\right)\right)^{(p-1)}(a) \ne \theta.$$

This means that a is a zero of multiplicity at least $p-1$ of each entry of $\operatorname{Adj}\left(P_n(z)\right)$. Hence a is a zero of multiplicity at least $p-1$ of $P(z)\operatorname{Adj}\left(P_n(z)\right)$ for any matrix polynomial P, and so, if $\operatorname{dgr}(P) \le n-1$, (*i.e.*, $\operatorname{dgr}\left(P\operatorname{Adj}(P_n)\right) \le nN-1 < \operatorname{dgr}\left(\det(P_n)\right)$), the expression (3.3) holds for

$$P(z)P_n^{-1}(z) = \frac{P(z)\operatorname{Adj}\left(P_n(z)\right)}{\det\left(P_n(z)\right)}.$$

We now prove (1) of Theorem 3.1. Let P be a matrix polynomial of degree less than or equal to $2n-1$. Since P_n is a polynomial with non-singular leading coefficient, we can write ([G] p. 78)

$$P(t) = C(t)P_n(t) + R(t), \tag{3.4}$$

where $C(t)$ and $R(t)$ are matrix polynomials with $\operatorname{dgr}(R) \leq n-1$.

Then we have

$$P(t)P_n^{-1}(t) = C(t) + R(t)P_n^{-1}(t), \tag{3.5}$$

when t is not a zero of P_n. Since $\operatorname{dgr}\big(R(t)\big) \leq n-1$ we can write

$$R(t)P_n^{-1}(t) = \sum_{k=1}^{m} C_{n,k}\frac{1}{t-x_{n,k}},$$

where the matrices $C_{n,k}$ are given by

$$C_{n,k} = \frac{1}{\Big(\det\big(P_n(t)\big)\Big)^{(l_k)}(x_{n,k})} R(x_{n,k})\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}).$$

Since $P_n(x_{n,k})\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}) = \theta$ (see part (4) of Theorem 2.3), (3.4) gives that the matrices $C_{n,k}$ can be written in the following way

$$C_{n,k} = \frac{1}{\Big(\det\big(P_n(t)\big)\Big)^{(l_k)}(x_{n,k})} P(x_{n,k})\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}). \tag{3.6}$$

Then (3.5) gives

$$P(t) = C(t)P_n(t) + \sum_{k=1}^{m} C_{n,k}\frac{P_n(t)}{t-x_{n,k}}.$$

From (3.6) and part (4) of Theorem 2.3, we have

$$P(t) = C(t)P_n(t) + \sum_{k=1}^{m} C_{n,k}\frac{P_n(t)-P_n(x_{n,k})}{t-x_{n,k}}.$$

From the definition of the polynomial Q_n it follows that

$$\int P(t)\,dW(t) = \int C(t)P_n(t)\,dW(t) + \sum_{k=1}^{m} C_{n,k}Q_n(x_{n,k}).$$

But $\operatorname{dgr}(C) < n$, so the orthonormality of P_n gives that $\int C(t)P_n(t)\,dW(t) = \theta$. Hence, we have

$$\int P(t)\,dW(t) = \sum_{k=1}^{m} C_{n,k}Q_n(x_{n,k}).$$

Then, part (1) of Theorem 3.1 follows from the definition of the matrices $\Gamma_{n,k}$ and $C_{n,k}$ (see (3.1) and (3.6)).

We now prove part (2) of Theorem 3.1:

We proceed in several steps:

STEP 1. The matrices $\Gamma_{n,k}$ ($k = 1, \ldots, m$) are hermitian.

PROOF. If we prove that the matrix $P_n^{-1}(t)Q_n(t)$ is hermitian when $t \in \mathbb{R}$ is not a zero of P_n, Step 1 would follow from the decomposition

$$P_n^{-1}(t)Q_n(t) = \sum_{k=1}^{m} \Gamma_{n,k} \frac{1}{t - x_{n,k}}.$$

The orthogonality of P_n with respect to any other polynomial of degree less than n gives that

$$\begin{aligned} P_n^{-1}(t)Q_n(t) &= \int P_n^{-1}(t) \frac{P_n(t) - P_n(x)}{t - x}\, dW(x) \\ &= \int P_n^{-1}(t) \frac{P_n(t) - P_n(x)}{t - x}\, dW(x)\big(P_n^*(t) - P_n^*(x)\big)(P_n^*)^{-1}(t) \\ &= \int P_n^{-1}(t)\big(P_n(t) - P_n(x)\big)\, dW(x) \frac{P_n^*(t) - P_n^*(x)}{t - x}(P_n^*)^{-1}(t) \\ &= \int dW(x) \frac{P_n^*(t) - P_n^*(x)}{t - x}(P_n^*)^{-1}(t) \\ &= Q_n^*(t)(P_n^*)^{-1}(t), \end{aligned}$$

that is, $P_n^{-1}(t)Q_n(t)$ is hermitian when $t \in \mathbb{R}$ is not a zero of P_n. ∎

STEP 2. $\operatorname{rank}(\Gamma_{n,k}) = l_k$, $k = 1, \dots, m$.

PROOF. Theorem 2.3 gives that $\operatorname{rank}\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}) = l_k$. Since the left eigenvectors of $\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k})$ associated to 0 are left eigenvectors of $\Gamma_{n,k}$ associated to 0 (see (3.1)), it will be enough to prove that left eigenvectors of $\Gamma_{n,k}$ associated to 0 are left eigenvectors of $\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k})$ associated to 0.

Let u be for which $v = u\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}) \neq \theta$. Part (3) of Theorem 2.3 shows that $v \in L(x_{n,k}, P_n)$. Then the formula (2.6) gives

$$v\big(Q_n(x_{n,k})P_{n-1}^*(x_{n,k}) - P_n(x_{n,k})Q_{n-1}^*(x_{n,k})\big) = vQ_n(x_{n,k})P_{n-1}^*(x_{n,k}) = vA_n^{-1}.$$

Since the matrix A_n^{-1} is non-singular, we have that $vQ_n(x_{n,k}) \neq \theta$, and taking into account the definition of v and $\Gamma_{n,k}$, we obtain $u\Gamma_{n,k} \neq \theta$. And Step 2 is proved. ∎

We take vectors $v_1, \dots, v_{l_k}$ satisfying the following properties:

(1) v_i is a left eigenvector of the matrix $\Gamma_{n,k}$ associated to an eigenvalue $\alpha_i \neq 0$, $i = 1, \dots, l_k$.

(2) $v_i v_j^* = \delta_{i,j}$, $i,j = 1, \dots, l_k$.

STEP 3. For $i = 1, \dots, l_k$, constants $\beta_i \neq 0$ exist such that

$$v_i\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}) = \beta_i v_i P_{n-1}^*(x_{n,k})A_n.$$

PROOF. From part (3) of Theorem 2.3, we have that $v_i\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}) \in L(x_{n,k}, P_n)$. Since the matrix $P_{n-1}^*(x_{n,k})A_n$ defines an isomorphism from $R(x_{n,k}, P_n)$ into $L(x_{n,k}, P_n)$ (Theorem 2.3 (2)), we have that

$$v_i\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k}) = u_i P_{n-1}^*(x_{n,k})A_n, \tag{3.7}$$

for certain $u_i \in R(x_{n,k}, P_n)$. Since $Q_n(x_{n,k})$ is the inverse mapping of $P^*_{n-1}(x_{n,k})A_n$, we have from (3.7) that

$$v_i\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k})Q_n(x_{n,k}) = u_i.$$

Then, from the definition of the matrix $\Gamma_{n,k}$ (see (3.1)), we deduce that

$$v_i\Gamma_{n,k} = \frac{1}{\big(\det P_n(t)\big)^{(l_k)}(x_{n,k})}u_i.$$

Since $v_i\Gamma_{n,k} = \alpha_i v_i$, it is enough to take $\beta_i = \alpha_i\big(\det P_n(t)\big)^{(l_k)}(x_{n,k})$. ■

Finally, we prove that $v_i\Gamma_{n,k}v_i^* > 0$, *i.e.*, the matrix $\Gamma_{n,k}$ is positive semidefinite. In Step 3, we have proved that $v_i = \frac{1}{\beta_i}u_i$, for certain $u_i \in R(x_{n,k}, P_n)$, then we deduce that $v_i \in R(x_{n,k}, P_n)$. From the formula

$$\begin{aligned}\Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k)}(x_{n,k})P_n(x_{n,k}) + \Big(\operatorname{Adj}\big(P_n(t)\big)\Big)^{(l_k-1)}(x_{n,k})P_n'(x_{n,k})&\\ = \big(\det P_n(t)\big)^{(l_k)}(x_{n,k})I,&\end{aligned}$$

and Step 3, we obtain

$$\beta_i v_i P^*_{n-1}(x_{n,k})A_nP_n'(x_{n,k})v_i^* = \big(\det P_n(t)\big)^{(l_k)}(x_{n,k}). \tag{3.8}$$

Step 3, (3.8), (2.3) and (2.5) give that

$$\begin{aligned}v_i\Gamma_{n,k}v_i^* &= \frac{v_i\Big(\operatorname{Adj}\big(P_n(x_{n,k})\big)\Big)^{(l_k-1)}Q_n(x_{n,k})v_i^*}{\big(\det P_n(t)\big)^{(l_k)}(x_{n,k})}\\ &= \frac{\beta_i v_i P^*_{n-1}(x_{n,k})A_nQ_n(x_{n,k})v_i^*}{\beta_i v_i P^*_{n-1}(x_{n,k})A_nP_n'(x_{n,k})v_i^*}\\ &= \frac{v_i\big(P^*_{n-1}(x_{n,k})A_nQ_n(x_{n,k}) - P^*_n(x_{n,k})A^*_nQ_{n-1}(x_{n,k})\big)v_i^*}{v_i\big(P^*_{n-1}(x_{n,k})A_nP_n'(x_{n,k}) - P^*_n(x_{n,k})A^*_nP'_{n-1}(x_{n,k})\big)v_i^*}\\ &= \frac{1}{v_i\big(\sum_{j=0}^{n-1}P_j^*(x_{n,k})P_j(x_{n,k})\big)v_i^*} > 0.\end{aligned}$$

And the theorem is proved. ■

Theorem 3.1 can also be extended for perturbations of the orthonormal polynomials. Indeed, let A be a matrix satisfying that A_nA is hermitian, and let us consider the polynomial $P_n(t) - AP_{n-1}(t)$. Then, Theorem 3.1 holds for this polynomial, *i.e.*:

THEOREM 3.2. *Let n be a nonnegative integer. We write $x_{n,k}$ ($k = 1, \dots, m$) for the different zeros of the matrix polynomial $P_n - AP_{n-1}$ (where $A_nA = A^*A_n^*$) ordered in increasing size (hence $m \le nN$) and $\Gamma_{n,k}$ for the matrices*

$$\begin{aligned}\Gamma_{n,k} = &\frac{1}{\Big(\det\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(l_k)}(x_{n,k})}\Big(\operatorname{Adj}\big(P_n(t) - AP_{n-1}(t)\big)\Big)^{(l_k-1)}(x_{n,k})\\ &\cdot\big(Q_n(x_{n,k}) - AQ_{n-1}(x_{n,k})\big), \quad k = 1, \dots, m,\end{aligned}$$

where l_k is the multiplicity of $x_{n,k}$.

(1) For any polynomial P with $\operatorname{dgr}(P) \leq 2n-2$ *the following formula holds*

$$\int P(t)\,dW(t) = \sum_{k=1}^{m} P(x_{n,k})\Gamma_{n,k}. \tag{3.9}$$

(2) The matrices $\Gamma_{n,k}$ are positive semidefinite matrices of rank l_k, for $k = 1, \dots, m$.

Using Theorem 2.4 instead of Theorem 2.3 the proof of Theorem 3.2 exactly works as that of Theorem 3.1. It is worth noting that (3.9) can not be extended for $\operatorname{dgr}(P) \leq 2n-1$ because of

$$\int C(t)\big(P_n(t) - AP_{n-1}(t)\big)\,dW(t) = \theta$$

can only be guaranteed when $\operatorname{dgr}(C) < n-1$.

REMARK 3.3. In the scalar case, the coefficients of the quadrature formula satisfy a remarkable property called Markov-Stieltjes inequalities:

$$\sum_{j=1}^{k-1} \lambda_{n,j} < \int_a^{x_{n,k}} d\mu(x) < \sum_{j=1}^{k} \lambda_{n,j}, \quad k = 1, \dots, n,$$

where $x_{n,1}, \dots, x_{n,n}$ are the zeros of the orthonormal polynomial p_n, and $\lambda_{n,k}$ are the quadrature weights given by

$$\lambda_{n,k} = \frac{1}{\sum_{j=1}^{n-1} |p_j(x_{n,k})|^2}.$$

This property was conjectured by Chebyshev in 1874, and proved independently by Markov and Stieltjes ten years later. We present here a counterexample showing that these Markov-Stieltjes inequalities are no longer true for orthogonal matrix polynomials. The reason is the highly singular character of quadrature weights in the matrix case.

Indeed, let μ_1, μ_2, $(p_{1,n})_n$ and $(p_{2,n})_n$ be two positive measures and their sequences of orthonormal polynomials. We define the matrix of measures W by

$$W = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Then it is clear that its sequence of orthonormal matrix polynomials is

$$P_n(t) = \begin{pmatrix} p_{1,n} & 0 \\ 0 & p_{2,n} \end{pmatrix}, \quad n \geq 0.$$

We assume that the measures μ_1 and μ_2 have the same support $[a,b]$ and that $p_{1,n}$ and $p_{2,n}$ interlace their zeros. Then an easy computation gives that the coefficients in the matrix quadrature formula are

$$\Gamma_{n,k} = \begin{cases} \begin{pmatrix} \lambda_{1,n,\frac{k+1}{2}} & 0 \\ 0 & 0 \end{pmatrix} & \text{for } k = 1, 3, \cdots, 2n-1, \\ \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{2,n,\frac{k}{2}} \end{pmatrix} & \text{for } k = 2, 4, \cdots, 2n, \end{cases}$$

where $\lambda_{1,n,k}$, $\lambda_{2,n,k}$, $(k = 1, \dots, n)$ are the quadrature weights for the measures μ_1, μ_2 respectively. Then, since

$$\Gamma_{n,1} = \begin{pmatrix} \lambda_{1,n,1} & 0 \\ 0 & 0 \end{pmatrix},$$

is a singular matrix, it is clear that

$$\int_a^{x_{1,n,1}} dW(x) \not\leq \Gamma_{n,1},$$

because of the non-singularity of $\int_a^{x_{1,n,1}} dW(x)$.

4. **Markov's theorem for orthogonal matrix polynomials.** We are now ready to prove the matrix version of Markov's theorem

PROOF OF THEOREM 1.1. First of all, we prove that

$$\lim_{n \to \infty} P_n^{-1}(z) Q_n(z) = \int \frac{dW(t)}{z-t}$$

for $z \in \mathbb{C} \setminus \Gamma$ (see (1.1) for the definition of Γ).

We consider the sequence of discrete matrices of measures $(\mu_n)_n$ defined by

$$\mu_n = \sum_{k=1}^{m} \delta_{x_{n,k}} \Gamma_{n,k}, \quad n \geq 0,$$

where $x_{n,k}$ $(k = 1, \dots, m)$ are the different zeros of the matrix polynomial P_n ordered in increasing size, and $\Gamma_{n,k}$ are the positive semidefinite matrices defined by (3.1) and which appear in the quadrature formula (3.2).

From the definition of these matrices (see (3.1)), it follows straightforwardly that

$$P_n^{-1}(z) Q_n(z) = \sum_{k=1}^{m} \Gamma_{n,k} \frac{1}{z - x_{n,k}} = \int \frac{d\mu_n(t)}{z-t}, \tag{4.1}$$

if z is not a zero of P_n. Taking into account (4.1), it will be enough to prove that

$$\lim_{n \to \infty} \int \frac{d\mu_n(t)}{z-t} = \int \frac{dW(t)}{z-t}$$

for $z \in \mathbb{C} \setminus \Gamma$. If not, we find a complex number $z \in \mathbb{C} \setminus \Gamma$, an increasing sequence of nonnegative integers $(n_m)_m$ and a positive constant C for which

$$\left\| \int \frac{dW(t)}{z-t} - \int \frac{d\mu_{n_m}(t)}{z-t} \right\|_2 \geq C > 0, \quad m \geq 0, \tag{4.2}$$

where we write $\| \cdot \|_2$ for the spectral norm of a matrix.

If we proceed as in Section 2 of [DL], taking two increasing sequences $(a_k)_k$, $(b_k)_k$ for which $a_k, b_k \longrightarrow +\infty$, we can obtain (by using Banach-Alaoglu's theorem) a subsequence $(l_m)_m$ from $(n_m)_m$, and a positive definite matrix of measures ν such that for $k \geq 0$

$$\lim_{m \to \infty} \int_{-a_k}^{b_k} f(t)\, d\mu_{l_m}(t) = \int_{-a_k}^{b_k} f(t)\, d\nu(t), \tag{4.3}$$

for any continuous matrix function f defined in $[-a_k, b_k]$. Now it is easy to prove (see Section 2 of [DL] for more details) that the k-th moment of the matrix of measures ν is the limit of the k-th moment of the matrices of measures μ_{l_m}. But the quadrature formula (3.2) gives that the k-th moment of the matrix of measures μ_{l_m} ($k \leq 2l_m - 1$) is precisely the k-th moment of the matrix of measures W. Since W is a determinate matrix of measures, we conclude that $\nu = W$. Then (4.3) gives that for $k \geq 0$

$$\lim_{m\to\infty} \int_{-a_k}^{b_k} f(t)\, d\mu_{l_m}(t) = \int_{-a_k}^{b_k} f(t)\, dW(t). \tag{4.4}$$

By taking k and then l_m big enough, from (4.2) and (4.4) we obtain

$$\frac{C}{2} \leq \left\| \int_{-\infty}^{-a_k} \frac{d\mu_{l_m}(t)}{z-t} + \int_{b_k}^{+\infty} \frac{d\mu_{l_m}(t)}{z-t} \right\|_2. \tag{4.5}$$

We write S_0 for the first moment of the matrices of measures μ_{l_m} which is the first moment of the matrix of measures W. Then we have from (4.5) and the definition of the spectral norm that

$$\begin{aligned}
\frac{C}{2} &\leq \max\left(\frac{1}{|z-a_k|}, \frac{1}{|z-b_k|}\right) \left\| \int_{-\infty}^{-a_k} d\mu_{l_m}(t) + \int_{b_k}^{+\infty} d\mu_{l_m}(t) \right\|_2 \\
&\leq \max\left(\frac{1}{|z-a_k|}, \frac{1}{|z-b_k|}\right) \left\| \int_{\mathbb{R}} d\mu_{l_m}(t) \right\|_2 \\
&\leq \max\left(\frac{1}{|z-a_k|}, \frac{1}{|z-b_k|}\right) \left\| S_0 \right\|_2.
\end{aligned}$$

But this implies $C = 0$, and therefore, (4.2) is not possible.

We now prove that the analytic functions which form the entries of the matrix $\int \frac{d\mu_n(t)}{z-t}$ are uniformly bounded in compact sets of $\mathbb{C} \setminus \Gamma$. Then, the uniform convergence in compact set of $\mathbb{C} \setminus \Gamma$ will follow from Stieltjes-Vitali's theorem.

Given a compact $K \subset \mathbb{C} \setminus \Gamma$, we notice that $K \cap M_N = \emptyset$, for N sufficiently big, and then there exists $C > 0$ such that

$$\left| \frac{1}{z-t} \right| \leq C \quad \text{for } z \in K \text{ and } t \in M_N.$$

Then, for $n \geq N$ and $v \in \mathbb{C}^N$, the positive definiteness of μ_n gives that

$$\left| v \int \frac{d\mu_n(t)}{z-t} v^* \right| = \int \frac{v\, d\mu_n(t) v^*}{|z-t|} \leq C \int v\, d\mu_n(t) v^* = v S_0 v^*,$$

and now it is easy to finish. ∎

It is worth to note that the order in which the polynomials P_n^{-1} and Q_n appear multiplied in the matrix version of Markov's theorem, *i.e.*, $P_n^{-1}(z)Q_n(z)$, is essential to guarentee the validity of this result. Indeed, let W be a positive definite matrix of measures and $(P_n)_n$, $(Q_n)_n$ its orthonormal matrix polynomials and polynomials of the second kind, respectively. Let us consider a non-singular matrix C, and the positive definite matrix of measures defined by $R = CWC^*$. It is clear that $(P_n C^{-1})_n$ and $(Q_n C^*)$ are, respectively, the sequence of orthonormal matrix polynomials and the sequence of polynomials of the second kind for R. Hence, if Markov's theorem held for $Q_n(z)P_n^{-1}(z)$, we would have

$$\lim_n Q_n(z)P_n^{-1}(z) = \int \frac{dW(t)}{z-t}$$
$$\lim_n Q_n(z)C^* CP_n^{-1}(z) = C\int \frac{dW(t)}{z-t}C^*,$$

which, in general, is clearly false.

Markov's theorem can be extended for perturbations of the orthonormal polynomials: Indeed, let A be a matrix satisfying that A_nA is hermitian, and let us consider the polynomial $P_n(t) - AP_{n-1}(t)$. We write $\Delta_{n,A}$ for the set of zeros of the matrix polynomial $P_n - AP_{n-1}$, and

$$\Gamma_A = \cap_{N\geq 0} M_{N,A}, \quad \text{where } M_{N,A} = \overline{\cup_{n\geq N}\Delta_{n,A}}.$$

Then

THEOREM 4.1. *Assume that W is determinate and that $A_nA = A^*A_n^*$ for $n \geq 0$. Then*

$$\lim_{n\to\infty}\big(P_n(z) - AP_{n-1}(z)\big)^{-1}\big(Q_n(z) - AQ_{n-1}(z)\big) = \int \frac{dW(t)}{z-t} \quad \textit{for } z\in \mathbb{C}\setminus\Gamma,$$

and the convergence is uniform for z in compact subsets of $\mathbb{C}\setminus\Gamma_A$.

Using Theorem 3.2 instead of Theorem 3.1 the proof of Theorem 4.1 exactly works as that of Theorem 1.1.

5. **The k-th associated polynomials.**

For $k \geq 1$, the k-th associated polynomials $(P_n^{[k]})_n$ are defined by the formula

$$P_n^{[k]}(x) = \int \frac{P_{n+k}(x) - P_{n+k}(t)}{x-t}\, dW(t)P_{k-1}^*(t), \quad n \geq 0. \tag{5.1}$$

The orthogonality of the polynomials $(P_n)_n$ shows that the degree of $P_n^{[k]}$ is just n.

Associated polynomials already appear in Stieltjes' fundamental work [S]. (See the survey [V] for more details).

These k-th associated polynomials satisfy the following two recurrence formulae

$$xP_n^{[k]}(x) = A_{n+k+1}P_{n+1}^{[k]}(x) + B_{n+k}P_n^{[k]}(x) + A_{n+k}^*P_{n-1}^{[k]}(x), \quad n \geq 0, \tag{5.2}$$

and

$$xP_n^{[k]}(x) = P_{n-1}^{[k+1]}(x)A_k^* + P_n^{[k]}(x)B_{k-1}^* + P_{n+1}^{[k-1]}(x)A_{k-1} \tag{5.3}$$

with initial condition $A_0 = I$, $P_{-1}^{[k]}(x) = \theta$ and $P_0^{[k]}(x) = A_k^{-1}$. Indeed, we have

$$xP_n^{[k]}(x) = \int \frac{xP_{n+k}(x) - tP_{n+k}(t)}{x-t}\, dW(t)P_{k-1}^*(t)$$
$$- \int \frac{(x-t)P_{n+k}(t)}{x-t}\, dW(t)P_{k-1}^*(t),$$

but also

$$xP_n^{[k]}(x) = \int \frac{P_{n+k}(x) - P_{n+k}(t)}{x - t}\, dW(t) t P_{k-1}^*(t) + \int \frac{P_{n+k}(x) - P_{n+k}(t)}{x - t}\, dW(t)(x - t) P_{k-1}^*(t).$$

Formulae (5.2) and (5.3) now follow from the three term recurrence formula for $(P_n)_n$ and the orthogonality of these polynomials.

From the formula (5.3) and the three term recurrence formula for $(P_n)_n$ one obtains the following *quasi* Christoffel-Darboux formula

THEOREM 5.1. *The following formulae are valid:*

$$(x - t) \sum_{k=1}^{n} P_{n-k}^{[k]}(x) P_{k-1}(t) = P_n(x) - P_n(t).$$

and

$$\sum_{k=1}^{n} P_{n-k}^{[k]}(x) P_{k-1}(x) = P_n'(x).$$

PROOF. Indeed, it is enough to write

$$\frac{P_n(x) - P_n(t)}{(x - t)} = \sum_{k=1}^{n} A_{k,n} P_{k-1}(t),$$

and compute the matrix coefficients $A_{k,n}$ by using the orthonormality of $(P_n)_n$. ∎

The following generalization of Markov's Theorem is a consequence of Theorem 1.1, the formula (5.3) and the three term recurrence formula for $(P_n)_n$ (see [V] for the scalar version):

THEOREM 5.2. *Assume that W is determinate. Then for $k \geq 1$*

$$\lim_{n \to \infty} P_n^{-1}(z) P_{n-k}^{[k]}(z) = \int dW(t) \frac{P_{k-1}^*(t)}{z - t} \quad \textit{for } z \in \mathbb{C} \setminus \Gamma,$$

and the convergence is uniform for z in compact subsets of $\mathbb{C} \setminus \Gamma$.

PROOF. It follows in a straightforward manner proceeding by induction on k and using the formula (5.3). ∎

REFERENCES

[AN] A. I. Aptekarev and E. M. Nikishin, *The scattering problem for a discrete Sturm-Liouville operator*, Mat. Sb. **121**(1983), 327–358; Math. USSR-Sb. **49**(1984), 325–355.

[B] C. Berg, *Markov's theorem revisited*, J. Approx. Theory **78**(1994), 260–275.

[D1] A. J. Duran, *A generalization of Favard's Theorem for polynomials satisfying a recurrence relation*, J. Approx. Theory **74**(1993), 83–109.

[D2] ———, *On orthogonal polynomials with respect to a positive definite matrix of measures*, Canad. J. Math. **47**(1995), 88–112.

[DL] A. J. Duran and P. Lopez-Rodriguez, *Orthogonal matrix polynomials: Zeros and Blumenthal's theorem*, J. Approx. Theory **84**(1996), 96–118.

[DV] A. J. Duran and W. van Assche, *Orthogonal matrix polynomials and higher order recurrence relations*, Linear Algebra Appl. **219**(1995), 261–280.

[G] F. R. Gantmacher, *The theory of matrices*, Chelsea Publishing Company, New York, **1**(1960).

[M] A. Markov, *Deux démonstrations de la convergence de certaines fractions continues*, Acta Math. **19**(1895), 93–104.

[SV] A. Sinap and W. van Assche, *Polynomial interpolation and Gaussian quadrature for matrix valued function*, Linear Algebra Appl. **207**(1994), 71–114.

[S] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Math. **8**(1894), 1–22; **[9]** (1895), 5–47.

[Z] D. Zhani, *Probleme des moments matriciels sur la droite: construction d'une famille de solutions et questions d'unicite*, Publications du Departement de Mathematiques, Universite Claude-Bernard-Lyon I, Nouvelle serie **D**(1984), 1–84.

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A CONSTRUCTIVE BRAUER-WITT THEOREM FOR CERTAIN SOLVABLE GROUPS

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ABSTRACT. Division algebras occurring in simple components of group algebras of finite groups over algebraic number fields are studied. First, well-known restrictions are presented for the structure of a group that arises once no further Clifford Theory reductions are possible. For groups with these properties, a character-theoretic condition is given that forces the p-part of the division algebra part of this simple component to be generated by a predetermined p-quasi-elementary subgroup of the group, for any prime integer p. This is effectively a constructive Brauer-Witt Theorem for groups satisfying this condition. It is then shown that it is possible to constructively compute the Schur index of a simple component of the group algebra of a finite nilpotent-by-abelian group using the above reduction and an algorithm for computing Schur indices of simple algebras generated by finite metabelian groups.

Let G be a finite group, $\mathbf{k}$ an algebraic number field, and let $\mathbf{k}G$ be the group algebra of the group G over $\mathbf{k}$. By the Artin-Wedderburn Theorem [CR, 3.28], we know that $\mathbf{k}G$ is semisimple. A simple component of $\mathbf{k}G$—one of its non-zero two-sided ideals—is always a principal ideal of the form $\mathbf{k}Ge$, where e is a centrally primitive idempotent of $\mathbf{k}G$. Being simple, $\mathbf{k}Ge$ is isomorphic to a matrix ring over a division algebra D finite-dimensional over $\mathbf{k}$. The square root of the dimension of D over its center is known as the *Schur index* of D over $\mathbf{k}$.

A standard result for computing the Schur index in the above situation in the case of a general finite group is the Brauer-Witt Theorem [W] (see also [Y, page 31]), stated in Section 2 of this paper. For each prime integer p, this theorem indicates the existence of a p-quasi-elementary section of G—see Definition 2 of this paper—that determines the p-part of the Schur index related to a given simple component of $\mathbf{Q}G$. (Recently, Schmid [Sch] has extended this theorem by identifying precise types of p-quasi-elementary groups that are the minimal groups one can reduce to using this theorem.) Complete descriptions of methods for the computation of Schur indices resulting from p-quasi-elementary groups have appeared in [Y], [L], and [Her], the last using an approach based on a paper of Janusz [J]. The methods used for these Schur index calculations rely heavily on number theoretic information because of the local norm calculations required for computing the index. However, the Brauer-Witt Theorem does not lend itself to an algorithmic method

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of determining sections of G suitable for computing the Schur index, so the methods available for computing Schur indices for general groups are not very practical.

There are several well-known theorems that indicate restrictions on the values of these Schur indices [I, Chap. 10]. Occasionally, group structure alone results in restrictions. The most notable of these restrictions is Roquette's Theorem that indicates Schur indices are at most 2 for nilpotent groups [Roq]. In addition to Roquette's Theorem, constructive computations of Schur indices that result from restrictions on the class of the finite group have appeared previously. Fontaine has demonstrated that Schur indices in the case of finite supersolvable groups can always be constructively computed using the Clifford theory reductions described in Section 1 of this paper [F, Prop. 3.4]. Janusz attained a constructive reduction for the Schur index question in the case where G has a normal subgroup N for which G/N is abelian using methods that involve projective representations [J2].

The present paper was motivated by the paper of Shirvani [Sh] which attained structural invariants for the metabelian case, and the author's work in [Her]. The main results of this paper are the following:

(1) For finite solvable groups G, there is a character-theoretic condition such that for any faithful irreducible character of G satisfying this condition, the Schur index of the simple component of $\mathbf{k}G$ corresponding to this character can be constructively computed using pre-determined subgroups of G (Theorem 4).

(2) The Schur indices of simple components of the group algebras of nilpotent-by-abelian groups over algebraic number fields can be constructively computed from a knowledge of the multiplication table of the group, using a reduction based on the above result (Section 3).

In order to deal constructively with simple components of $\mathbf{k}G$, we develop the following somewhat standard notation, based on based on [I] and [Y]. Let $\operatorname{Irr}(G)$ denote the set of irreducible complex characters of G. For any $\chi \in \operatorname{Irr}(G)$, let

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(1)\chi(g^{-1})g$$

denote the centrally primitive idempotent of $\mathbb{C}G$ determined by χ. It is well known that the simple component $\mathbb{C}Ge_\chi$ is a G-module affording the character $\chi(1)\chi$ [Y, Chapter 1]. For any subfield $\mathbf{k}$ of $\mathbb{C}$, let

$$a_\chi = \sum_{\sigma \in \mathfrak{G}} e_\chi^\sigma,$$

where $\mathfrak{G} = \operatorname{Gal}(\mathbf{k}(\chi)/\mathbf{k})$. Then a_χ is a centrally primitive idempotent of $\mathbf{k}G$, and the simple component $\mathbf{k}Ga_\chi$ of $\mathbf{k}G$ affords the character $\chi(1)\sum_{\sigma\in\mathfrak{G}} \chi^\sigma$ as a G-module.

$\mathbf{k}Ga_\chi$ is isomorphic to a ring of $n \times n$ matrices over a division algebra D, for some positive integer n. This division algebra D is finite dimensional over its center $\mathbf{k}(\chi)$, the field of character values of χ over $\mathbf{k}$ [I, Exercise 9.15]. Furthermore, the integer $m_{\mathbf{k}}(\chi)$ for which

$$|D\colon \mathbf{k}(\chi)| = m_{\mathbf{k}}(\chi)^2$$

is the Schur index of χ (or D) over $\mathbf{k}$ [CR, page 585]. In what follows, the goal is to find a constructive means of computing the Schur index $m_{\mathbf{k}}(\chi)$ that works for as large a class of groups as possible.

1. **Clifford Reductions.** Assume now that χ is a faithful character of the finite group G. Clifford's Theorem [I,6.2] implies that whenever $N \lhd G$ and the subalgebra $\mathbf{k}Ne_\chi$ of $\mathbf{k}Ge_\chi$ is not simple, then

$$\mathbf{k}Ge_\chi \cong (\mathbf{k}Ca_\lambda e_\chi)^{k\times k},$$

where C is the centralizer in the group G of a centrally primitive idempotent a_λ of $\mathbf{k}Ne_\chi$, and $k = |G : C|$ [P, 6.1.7]. The idempotent a_λ is the central idempotent of $\mathbf{k}N$ determined by some fixed irreducible constituent λ of χ_N. Furthermore, it follows from [I, 6.11] and the fact that C always contains the stabilizer of the character λ that $\chi = \psi^G$, for some irreducible character ψ of C lying over λ, and so we even know that $e_\chi = a_\psi$. Thus the division algebra part of $\mathbf{k}Ce_\chi$ is isomorphic to that of $\mathbf{k}Ge_\chi$, and so we might as well replace G by C and χ by ψ. In order to maintain our original assumption that χ is faithful, we can further replace G by $C/(\ker\psi)$.

This reduction of the Schur index problem to one concerning a smaller group is constructive, because the subgroup $C = C_G(a_\lambda)$ can be obtained (although with possibly quite a bit of work) from the multiplication table of G and from some familiarity with $\mathbf{k}$. One first computes the character tables of G and N and then uses the formula to write down the idempotents e_χ and all of the e_λ, $\lambda \in \mathrm{Irr}(N)$. Once one has found a λ for which $e_\chi e_\lambda$ is non-zero, one can use the character values of λ and the field $\mathbf{k}$ to determine a_λ. Then one can determine C by finding the centralizer in G of a_λ. Finally, ψ is obtained by using the character table of C to find an idempotent e_ψ such that $e_\chi e_\psi \neq 0$, $e_\lambda e_\psi \neq 0$, and $\chi(1) = |G : C|\psi(1)$.

Only so many non-trivial Clifford reductions of the above type are possible, of course, and so eventually one is reduced to the case where $\mathbf{k}Ne_\chi$ is a simple algebra, for every normal subgroup N of G. We will refer to a group that has a faithful character for which $\mathbf{k}Ga_\chi$ has this property as a *Clifford reduced* group over $\mathbf{k}$. The structure of Clifford reduced groups can be partly inferred from the structure of finite groups with the property that every normal abelian subgroup is cyclic, because Clifford reduced groups automatically have this property. Very precise structural information is available for these groups.

PROPOSITION. *[Hup, III.13.10] Suppose that G is a finite group such that every normal abelian subgroup of G is cyclic. Then $F(G)$ has a characteristic subgroup F of index at most 2. This subgroup F is the central product of a cyclic group $U = Z(F)$ with a group E that is the direct product of extraspecial p-groups of exponent p or 4, at most one extraspecial p-group for each prime integer p dividing the order of F. In particular, F/U is elementary abelian. Always, we have $F = C_{F(G)}\Big(\Phi\big(F(G)\big)\Big)$.*

If we further suppose that G is solvable, then further restrictions can be obtained.

LEMMA. *Assume that the finite group G is solvable, and that every normal abelian subgroup of G is cyclic. Let $F = C_{F(G)}\Big(\Phi\big(F(G)\big)\Big)$ be the characteristic subgroup of G given in the above. Then $Z(F) = C_G(F)$.*

PROOF. Let $U = Z(F)$. If $F = F(G)$, then since $C_G\big(F(G)\big) \subseteq F(G)$ when G is solvable [Sc, 7.4.7], we must have

$$U = C_G(F) \cap F = C_G(F),$$

and we are done.

If F has index 2 in $F(G)$, then note that the Fitting subgroup of $C_G(F)$ must be $F \cap C_G(F) = U$, because the Fitting subgroup of $C_G(F)$ is a normal nilpotent subgroup of G and hence lies inside F.

Suppose $C_G(F)$ properly contains U. Let X/U be a chief section of G with $X \subseteq C_G(F)$. Because G is solvable, X/U is a non-trivial p-group, for some prime p. Since U is the Fitting subgroup of $C_G(F)$, U is also the Fitting subgroup of X, and so X cannot centralize U. This contradicts the assumption that X centralizes F, and so we conclude that $U = C_G(F)$. ∎

The above lemma allows one to come to certain conclusions about the structure of certain sections of the group G.

PROPOSITION. *Let G be a solvable group, and suppose that every normal abelian subgroup of G is cyclic. Let $F = C_{F(G)}\Big(\Phi\big(F(G)\big)\Big)$. Then if we let $U = Z(F)$ and $C = C_G(U)$, we have that G/C is abelian, and C/F is isomorphic to a solvable subgroup of the direct product of finite symplectic groups of the form* $\mathrm{Sp}(2n, p)$, *for any prime integer p and power n such that $|F : U|$ is divisible by p^{2n}.*

2. **A Condition on Characters.** In this section, we assume that G is a finite solvable group that is Clifford reduced over an algebraic number field $\mathbf{k}$ with respect to a faithful irreducible character χ. Our goal is to find conditions which guarantee that a further constructive reduction of the Schur index problem is possible.

Fix a prime integer p throughout. The following definition will make it easier to state our results.

DEFINITION 1. Let χ be a faithful irreducible character of a finite group G, and let $\mathbf{k}$ be an algebraic number field for which $\mathbf{k} = \mathbf{k}(\chi)$. Let $e = \exp(G)$. The *p'-splitting field of $\mathbf{k}Ge_\chi$ over* $\mathbf{k}$ is the unique subextension K of $\mathbf{k}(\zeta_e)$ containing $\mathbf{k}$ such that

$$[\mathbf{k}(\zeta_e) : K] \text{ is a power of } p, \text{ and}$$
$$[K : \mathbf{k}] \text{ is relatively prime to } p.$$

Because cyclotomic extensions are always abelian Galois extensions when the base field has characteristic zero, it follows that the K defined above is the field fixed by the unique

Sylow p-subgroup of $\mathrm{Gal}(\mathbf{k}(\zeta_e)/\mathbf{k})$, and so K is unique. The property of this K that we need is that the Schur index of the division algebra part of

$$KGe_\chi \cong K \otimes_{\mathbf{k}} \mathbf{k}Ge_\chi$$

will be precisely the p-part of $m_{\mathbf{k}}(\chi)$.

In order to state the Brauer-Witt Theorem, we need the following definition.

DEFINITION 2. A finite group is called *p-quasi-elementary* if it is isomorphic to the split extension of a cyclic group of order relatively prime to p by a finite p-group.

THEOREM (BRAUER-WITT). *Let χ be an irreducible character of a finite group G. Let $\mathbf{k}$ be an algebraic number field satisfying $\mathbf{k}(\chi) = \mathbf{k}$. Let K be the p'-splitting field for $\mathbf{k}Ge_\chi$ over $\mathbf{k}$.*

Then there exists a p-quasi-elementary section H of G and a character $\xi \in \mathrm{Irr}(H)$ such that

$$m_K(\xi) = \left(m_{\mathbf{k}}(\chi)\right)_p.$$

Proofs of the Brauer-Witt Theorem are based on Brauer's Induction Theorem [CR, 15.9], which is used to establish the existence of the section H of the correct type and the desired character ξ [Y, Chapter 3]. This approach does not lead by itself to an algorithmic method of finding such a section H from a knowledge of the subgroup structure of G, and thus has limited practical applications. In what follows, we will show that the Brauer-Witt Theorem can be made constructive when χ satisfies a certain character-theoretic condition.

Assuming that the solvable group G is Clifford reduced over $\mathbf{k}$ with respect to a faithful irreducible character χ, let K be the p'-splitting field for $\mathbf{k}Ge_\chi$ over $\mathbf{k}$. Since the p-part of the Schur index of $\mathbf{k}Ge_\chi$ is exactly the Schur index of KGe_χ, we may replace $\mathbf{k}$ by K.

The first step is to re-do the Clifford Theory reductions of the previous chapter with respect to the new field K. Assume we have completed this process, so that for every normal subgroup N of G, KNe_χ is a simple algebra. In particular, every normal abelian subgroup of G is cyclic, so G has normal subgroups F, U, and C satisfying the following conditions of the previous section:

(1) F is a characteristic nilpotent subgroup of G having index at most 2 in $F(G)$;
(2) $U = Z(F)$ is cyclic, and F/U is elementary abelian;
(3) F is the central product of a group E with U, where E is a direct product of extraspecial q-groups having exponent q (or 4 when q is 2), for some prime integers q; and
(4) $C = C_G(U)$.

We now establish various character identities for irreducible characters of the above subgroups of G.

PROPOSITION 3. *Let K be the p'-splitting field for the simple algebra $\mathbf{k}Ge_\chi$. Suppose that KGe_χ has been fully reduced using Clifford theory, and let F, U, and C be the subgroups of G defined above. Let S be a transversal of C in G. Then*

(i) $\chi_U = k\sum_{s\in S}\lambda^s$, *for some faithful irreducible character* λ *of* U *and some integer* $k > 0$;

(ii) $G/C \cong \mathrm{Gal}\big(K(\lambda)/K\big)$;

(iii) $\chi = \psi^G$, *for some* $\psi \in \mathrm{Irr}(C)$;

(iv) $\chi_F = d\sum_{s\in S}\varphi^s$, *for some faithful irreducible character* φ *of* F, *and some integer* $d > 0$.

(v) φ *and* ψ *may be chosen so that* $\varphi_U = f\lambda$, *with* $f^2 = [F:U]$, *and* $\psi_F = d\varphi$;

(vi) $K(\lambda) = K(\varphi) = K(\psi) = K(\zeta_u)$, *where* $u = |U|$.

PROOF. Since $U \lhd G$, the algebra KUe_χ is simple under our assumptions. Thus

$$\chi_U = k\sum_{\sigma\in\mathcal{G}}\lambda^\sigma,$$

where λ is some irreducible character of U, $\mathcal{G} = \mathrm{Gal}\big(K(\lambda)/K\big)$, and k is some positive integer. Because all of the characters λ^σ lying under χ are Galois conjugate, all of their kernels are the same. Thus

$$\ker\lambda = \bigcap_{\sigma\in\mathcal{G}}\ker\lambda^\sigma = \ker\chi_U = U\cap\ker\chi = U\cap 1 = 1,$$

so λ is a faithful irreducible character of U. Since λ is a faithful linear character, the stabilizer of λ in G must exactly be the centralizer in G of the cyclic group U; namely, C. The character-theoretic version of Clifford's Theorem gives the decomposition

$$\chi_U = k\sum_{s\in S}\lambda^s,$$

where S is a transversal of C in G. This proves (i). (iii) also follows because χ is always induced from the stabilizer in this situation [I, 6.11]. Mapping G onto permutations on the set $\{\lambda^\sigma : \sigma\in\mathcal{G}\}$ gives a natural isomorphism of G/C with $\mathcal{G}$, namely $gC \longmapsto \sigma(g)$, where $\sigma(g)$ is defined by $\lambda^g = \lambda^{\sigma(g)}$. This proves (ii).

Now, $\chi = \psi^G$, for some $\psi \in \mathrm{Irr}(C)$ such that $\psi_U = k\lambda$. This forces $K(\lambda) \subseteq K(\psi)$. If $K(\lambda) \neq K(\psi)$, then there is a galois automorphism fixing λ but not ψ, and so there is more than one ψ lying over λ and inducing χ. This would contradict [I, 6.11(c)]. (This method originates in [C, Lemma 1.1].) Thus $K(\lambda) = K(\psi)$.

As in the case of U, $F \lhd G$ implies that

$$\chi_F = d\sum_{\tau\in\mathcal{H}}\varphi^\tau,$$

where φ is some irreducible character of F, $\mathcal{H} = \mathrm{Gal}\big(K(\varphi)/K\big)$, and d is some positive integer. As above, we can show that φ is faithful. Because F is nilpotent of class 2, it follows from [I, 2.31] that any faithful character of F vanishes off $Z(F) = U$. Thus we can choose $\varphi \in \mathrm{Irr}(F)$ so that $\varphi_U = f\lambda$, with $f^2 = [F : U]$. Since φ must vanish off U, we have that $K(\varphi) = K(\lambda)$, and the stabilizer of φ is exactly C. This proves (vi), and (iv) follows as in the above with λ replaced by φ.

As $\psi_F = d\varphi^s$, for some $s \in S$, the fact that $(\psi_F)_U$ is a multiple of λ implies that $\psi_F = d\varphi$. Statement (v) follows. ∎

One final remark is needed before the proof of the main result of this section. From our assumption concerning K, we see that by part (ii) of the above proposition, G/C must be a p-group. Thus for any Sylow p-subgroup of G, we have $G = CP$. Since non-identity elements of a transversal of C in G must lie outside C, we can choose the transversal S such that $S \subseteq P$, for any previously chosen Sylow p-subgroup P.

THEOREM 4. *Let K be the p'-splitting field for the simple algebra* $\mathrm{k}Ge_\chi$. *Suppose that KGe_χ has been fully reduced using Clifford theory, and let F, U, and C be the subgroups of G defined above. Let P be a Sylow p-subgroup of G for the fixed prime p. Suppose that*

$$H = UP = U_{p'} \rtimes P,$$

where $U_{p'}$ is the p-complement of the cyclic group U.

Then

(i) H is a p-quasi-elementary subgroup of G;

(ii) If $\psi_F = d\varphi$, with $(d,p) = 1$, then there exists a $\xi \in \mathrm{Irr}(H)$ such that $(\chi_H, \xi) \not\equiv 0 \bmod p$.

(iii) If $\psi_F = \varphi$, then there exists a $\xi \in \mathrm{Irr}(H)$ such that $(\chi_H, \xi) \not\equiv 0 \bmod p$ and $K(\xi) = K$.

PROOF. (i) is clear, since U is a cyclic normal subgroup of G.

To prove (ii), we let Q be the normal subgroup $H \cap F$ of G, and we compare expressions for the characters $(\chi_F)_Q$ and $(\chi_H)_Q$. Suppose that we have chosen a transversal S of C in G so that $S \subseteq P$. Then we have that

$$\begin{aligned}(\chi_F)_Q &= d\sum_{s\in S}(\varphi^s)_Q \\ &= \frac{df}{\sqrt{[Q:U]}}\sum_{s\in S}\theta^s,\end{aligned}$$

for some $\theta \in \mathrm{Irr}(Q)$ that is fully ramified with respect to U and lies over λ. On the other hand, suppose

$$\chi_H = \sum_{i=1}^{r} c_i\xi_i,$$

for some positive integers $c_1, \dots, c_r$, and irreducible characters $\xi_1, \dots, \xi_r$ of Q. Since the Galois conjugates of θ are the only irreducible characters of Q lying under χ, they are also the only irreducible characters of Q that lie under any of the characters $\xi_1, \dots, \xi_r$. Since $\{\theta^s : s \in S\}$ is exactly the set of Galois conjugates of θ, and each $s \in H$, we conclude that for each $i = 1, \dots, r$, we have

$$(\xi_i)_Q = p^{k_i}\sum_{s\in S}\theta^s,$$

for some integer $k_i \geq 0$. Therefore

$$\begin{aligned}(\chi_H)_Q &= \sum_{i=1}^{r} c_i(\xi_i)_Q, \\ &= \sum_{i=1}^{r} c_i(p^{k_i} \sum_{s\in S} \theta^s) \\ &= \sum_{i=1}^{r} c_i p^{k_i}(\sum_{s\in S} \theta^s).\end{aligned}$$

Comparing the two expressions for $\chi_Q = (\chi_F)_Q = (\chi_H)_Q$, we conclude that

$$\sum_{i=1}^{r} c_i p^{k_i} = \frac{df}{\sqrt{[Q:U]}}.$$

By our assumption that d is relatively prime to p, the second number has to be relatively prime to p. This implies that at least one of the numbers $c_i p^{k_i}$ in the sum on the left has to be relatively prime to p. Without loss of generality, assume $c_1 p^{k_1}$ is relatively prime to p. Then we must have $k_1 = 0$ and $c_1 = (\xi_1, \chi_H) \not\equiv 0 \bmod p$. This $\xi = \xi_1$ is the irreducible character of H required for (ii). (Note that we have also shown that for this character ξ,

$$\xi_Q = \sum_{s\in S} \theta^s.)$$

To prove (iii), note that the assumption that $\psi_F = \varphi$ is equivalent to

$$KFa_\varphi \cong KCa_\psi,$$

since $K(\psi) \cong K(\varphi)$ and both algebras have dimension $\psi(1)^2 = \varphi(1)^2$ over their centers. Our assumption on G then gives $KFe_\chi = KCe_\chi$, and so

$$KCe_\chi \cap KHe_\chi = KFe_\chi \cap KHe_\chi.$$

The subalgebra KHe_χ need not be simple because H need not be normal in G. However, we know from $\chi_H = \sum_i c_i \xi_i$ that

$$KHe_\chi = \oplus_i KHa_{\xi_i} e_\chi,$$

because KHe_χ is a homomorphic image of KH as an algebra. (The simple components of KH that occur in this sum will be exactly those whose associated irreducible characters are constituents of χ_H.) Let $\theta \in \mathrm{Irr}(Q)$ be as above. Since θ vanishes off U, it follows that $I_H(\theta) = H \cap C$. Let $R = H \cap C$, and note that FR/F is a Sylow p-subgroup of C/F.

Now let $\xi \in \mathrm{Irr}(H)$ be the character found in the proof of part (ii). From the remark that

$$\xi_H = \sum_{s\in S} \theta^s,$$

it follows from $R = I_H(\theta)$ that there is a character $\eta \in \mathrm{Irr}(R)$ such that $\xi = \eta^H$ and $\eta_Q = \theta$.

Consider the subalgebra KRe_χ. We have that $KRe_\chi \subseteq KCe_\chi$, so there are algebra inclusions

$$KQe_\chi \subseteq KRe_\chi \subseteq KFe_\chi.$$

Since $\varphi_Q = f\theta$ with $f = \sqrt{|F:Q|}$, we know that

$$[KFe_\chi : KQe_\chi] = |F:Q|$$

is relatively prime to p. Thus the index of KQe_χ in the subalgebra KRe_χ of KFe_χ has to be relatively prime to p. But R/Q is a p-group, and K is the p'-splitting field, so $[KRe_\chi : KQe_\chi]$ must be a power of p. This forces $KRe_\chi = KQe_\chi$.

Note that each of the characters ξ_i occurring in the above sum is faithful on Q by the formula $(\xi_i)_Q = p^{k_i} \sum_S \theta^s$. Since KQe_χ is a simple subalgebra of KHe_χ, this means that each of the maps $KQe_\chi \longrightarrow KQa_{\xi_i}e_\chi$ has to be an isomorphism. (Hence we can interpret the inclusion

$$KQe_\chi \longrightarrow \oplus_i KHa_{\xi_i}e_\chi$$

as a diagonal embedding.) Since $KRe_\chi = KQe_\chi$, we have that $KRa_\xi e_\chi = KQa_\xi e_\chi$ is isomorphic to the simple component KRa_η of KR. The center of KRa_η is an isomorphic copy of $K(\eta)$, and so since $KQe_\chi \cong KQa_\xi e_\chi \cong KRa_\eta$, we conclude that $K(\theta) = K(\eta)$.

Finally, note that since ξ is induced from $I_H(\theta) = R$, ξ vanishes off R. Since H/R acts as galois automorphisms on the subfield $K(\theta) = KUa_\xi e_\chi$, H/R acts as galois automorphisms on $K(\eta) = Z(KRa_\eta e_\chi)$. Thus $K(\xi) \cong Z(KHa_\xi e_\chi)$ is the subfield of $K(\theta)$ fixed by H/R. Of course, this is exactly K, as required. ∎

COROLLARY 4.1. *Under the conditions of Theorem 4, if we have* $\psi_F = \varphi$, *then* $\big(m_{\mathbf{k}}(\chi)\big)_p = m_{\mathbf{k}}(\xi)$.

PROOF. The conclusion of Theorem 4, part (iii) is exactly what is needed to apply [Y, Corollary 3.8], from which we get $m_K(\chi) = m_K(\xi)$. The corollary follows because Schur indices of characters of p-quasi-elementary groups have to be powers of p and $[K : \mathbf{k}]$ is prime to p, which forces $m_K(\xi) = m_{\mathbf{k}}(\xi)$ by [Rit, Theorem 2(c)]. ∎

We are now motivated to find conditions for the group G that ensure that the character φ of F extends to its stabilizer C once G has been reduced using Clifford theory. Exact conditions for this are unknown to this author, although a considerable amount of literature is available on conditions for character extension. (See, for instance, [G] and [I2]). Of course, if C/F is cyclic, then it is well known that φ extends to C [I, 11.22].

3. **The Nilpotent-by-Abelian Group Case.**

In this section, we assume that G is a finite nilpotent-by-abelian group that is Clifford reduced with respect to its faithful irreducible character χ. We conclude the paper by showing that for such a nilpotent-by-abelian group, the same p-quasi-elementary group H as in Theorem 4 above can be used to compute the Schur index of χ.

From the structure results in Section 1, we first note that the following holds for nilpotent-by-abelian groups.

PROPOSITION 5. *Suppose that G is a finite nilpotent-by-abelian group and $\chi \in \mathrm{Irr}(G)$. Then $\mathbf{k}Ge_\chi$ Clifford reduces to a simple component of the group algebra $\mathbf{k}\overline{G}$, where $\overline{G}$ is a nilpotent-by-abelian group of derived length at most 3.*

PROOF. Assume that $\mathbf{k}Ge_\chi$ has been fully reduced using Clifford theory, and let the subgroups F, U, and C be as above. Since the class of nilpotent-by-abelian groups is closed under taking sections, we still have that G is nilpotent-by-abelian. We know that $F = F(G) \cap C$ and G/C is abelian, so our assumption that $G/F(G)$ is abelian implies that

$$G' \subseteq F(G) \cap C = F,$$

and so G/F is abelian. The proposition follows because F is a nilpotent group of class at most 2. ∎

(We remark that if G is any solvable group, the Clifford theory reductions on $\mathbf{k}Ga_\chi$ end in a simple component of the group algebra of a group having derived length at most the derived length of $G/F(G)$ plus 3.)

Now, let K be the p'-splitting field for $\mathbf{k}Ge_\chi$, and suppose that KGe_χ has been fully reduced using Clifford Theory. Let F, U, and C be the characteristic subgroups of G determined in Section 1, and let $\varphi \in \mathrm{Irr}(F)$, $\lambda \in \mathrm{Irr}(U)$, and $\psi \in \mathrm{Irr}(C)$ satisfy the conclusions of Proposition 3. By the proof of Proposition 5 we know that G/F is abelian.

LEMMA 6. *Under the assumptions of the preceding paragraph, there exists a normal subgroup B and an irreducible character β of B such that*

(i) $F \subseteq B \subseteq C$;

(ii) $\chi = \beta^G$, $\psi = \beta^C$; *and*

(iii) $\beta_F = \varphi$.

PROOF. Because G/F is abelian, G is a relative M-group with respect to F [I, 6.22], which means that for every character $\chi' \in \mathrm{Irr}(G)$, there is a subgroup B' containing F and a $\beta \in \mathrm{Irr}(B')$ such that $\beta'^G = \chi'$ and $\beta'_F \in \mathrm{Irr}(F)$. Assuming B and β satisfy this for the character χ, we see that $\chi = \beta^G$. Since G/F is abelian, B has to be a normal subgroup of G. The assumption that KGe_χ is fully reduced using Clifford theory then implies that KBe_χ is simple, hence $KBe_\chi \cong KBa_\beta$. Since $\chi = \beta^G$, B is the stabilizer of β in G, and so

$$G/B \cong \mathrm{Gal}(K(\beta)/K).$$

Since the G-conjugates of λ are the only irreducible characters of F lying under χ, we must have $\beta_F = \varphi^g$, for some $g \in G$. This φ^g is thus invariant in B. Since C is the stabilizer of every φ^g in G, this implies that $B \subseteq C$. As $\beta^{g^{-1}}$ also induces χ, replacing β by $\beta^{g^{-1}}$ gives us the condition $\beta_F = \varphi$. Since β^C is irreducible and lies over φ, we get that $\beta^C = \psi$ because $\psi \in \mathrm{Irr}(C)$ is unique with this property. All three conclusions have now been shown. ∎

The next step is to reduce to a group having a normal p-complement.

LEMMA 7. *Suppose that G is a nilpotent-by-abelian group with faithful irreducible character χ. Assume KGe_χ has been fully reduced using Clifford theory, where K is the p'-splitting field of $\mathbb{Q}(\chi)Ge_\chi$. Let M be a subgroup of G so that M/F is the Sylow p-subgroup of G/F, F as above. Then $KMe_\chi = KGe_\chi$.*

PROOF. Let B be as in Lemma 6. Since $G/B \cong \mathrm{Gal}\big(K(\beta)/K\big)$ is a p-group, we can choose a transversal X of B in G consisting of elements of M. Since φ extends to $\beta \in \mathrm{Irr}(B)$, we see that

$$KBe_\chi \cong KFe_\chi \otimes_{K(\varphi)} K(\beta),$$

because $\beta(1) = \varphi(1)$. Let D be the subgroup of B for which D/F is a Sylow p-subgroup of B/F. Note that KDe_χ must be simple because $D \triangleleft G$. Since φ extends to D, the same reasoning as above shows that

$$KDe_\chi \cong KFe_\chi \otimes_{K(\varphi)} K(\beta_D).$$

Now we see that $[KBe_\chi : KDe_\chi] = [K(\beta) : K(\beta_D)]$, which is a power of p by our assumption on K. However, since $[KBe_\chi : KDe_\chi]$ divides $|B : D|$, it is relatively prime to p. We conclude that $KDe_\chi = KBe_\chi$. Since $\beta^G = \chi$, we have

$$\begin{aligned} KGe_\chi &= \oplus_{x\in X}(KBe_\chi)x \\ &= \oplus_{x\in X}(KDe_\chi)x \\ &= KMe_\chi, \end{aligned}$$

since X is also a transversal of D in M. ■

Using Lemma 7, we now assume G/F is a p-group by replacing G by M if necessary. If P is a Sylow p-subgroup of G, then we have that $G = FP$, and so since the p'-part of F is normal in G, G can be written as

$$G = N \rtimes P,$$

with the order of N being relatively prime to P. Together with the assumption that KGe_χ is fully reduced using Clifford theory, this presentation forces the p-part of the subgroup F to be cyclic. (This observation follows from [MW, Corollary 1.10 (iii)] and the fact that every chief section of such a group G that lies over N must have prime order p.) Suppose that B is the normal subgroup of G identified in Lemma 6 that possesses an irreducible character β extending φ and inducing χ. The next step is to arrange that the field of character values $K(\beta)$ is a cyclotomic extension of the p'- splitting field K.

Let $e = \exp(B)$. Since B/F is a p-group, we must have $e = p^c \exp(F)$, for a non-negative integer c. From our knowledge of the structure of F we find that $\exp(F)$ is either $u = |U|$ or $2u$. Furthermore, $\exp(F)$ can be $2u$ only when p is odd, 4 divides $|N|$, and 4 does not divide $|U|$. The inclusions

$$K(\varphi) = K(\zeta_u) \subseteq K(\beta) \subseteq K(\zeta_e)$$

are obvious, with both $[K(\zeta_e) : K(\beta)]$ and $[K(\beta) : K(\zeta_u)]$ equaling powers of the prime p. Since e is either $p^c u$ or $2p^c u$, we have that whenever p is an odd prime, the subfields of $K(\zeta_{p^c u})$ (or $K(\zeta_{p^c u}, \zeta_4)$ when $e = 2p^c u$) that have index a power of p over $K(\zeta_u)$ are all of the form $K(\zeta_{p^\ell u})$, for some integer ℓ such that $0 \leq \ell \leq c$. (This is immediate because the p-part of the respective Galois groups are cyclic.) So in the case where p is odd, $K(\beta) = K(\zeta_{p^\ell u})$ for some integer ℓ, and ℓ can be determined easily from e and $|G : B|$.

If $p = 2$, then the subfields of $K(\zeta_{2^c u})$ lying over $K(\zeta_u)$ do not have to be of the form $K(\zeta_{2^\ell u})$, for $1 \leq \ell \leq c$, unless 4 divides u. If 4 divides u, then we determine that $K(\beta)$ is a cyclotomic extension of K exactly as above. Assuming that 4 does not divide u, we "inflate" the group G to a new group, $\hat{G}$, and arrange that 4 divides the order of the necessary cyclic normal subgroup of $\hat{G}$. If $|U|$ is even, define $\hat{G}$ to be the central product of G with the abstract group

$$D_8 := \langle x, y \mid x^4 = 1 = y^2, x^y = x^{-1} \rangle$$

—a copy of the dihedral group of order 8. If $|U|$ is odd, then define $\hat{G}$ to be the direct product of G with the same group D_8. $K\hat{G}$ has a simple component that is isomorphic to the tensor product

$$KGe_\chi \otimes_K K^{2\times 2}.$$

This simple component is associated with a faithful representation of $\hat{G}$. Not every normal abelian subgroup of $\hat{G}$ is cyclic, since D_8 does not have this property. However, if we set $\hat{F} = \langle F, x \rangle$, $\hat{U} = \langle U, x \rangle$, and $\hat{B} = \langle B, x \rangle$, then any faithful character associated with the above simple component is induced from a faithful irreducible character $\hat{\beta}$ of $\hat{B}$. Since $K(\hat{\beta})$ is a subfield of $K(\zeta_{2^c u})$ containing $K(\zeta_u, \zeta_4)$, $K(\hat{\beta})$ has to be a cyclotomic extension of K, as required. Since the Schur index of the new simple component is the same as that of KGe_χ, it suffices to do our computations based on the slightly larger nilpotent-by-abelian group $\hat{G}$, with the above definitions for the corresponding subgroups and their characters.

Once we have that the character field $K(\beta)$ is a cyclotomic field, we can "inflate" the group G in order to apply Theorem 4 directly. (In the special case, replace G by $\hat{G}$, B by $\hat{B}$, β by $\hat{\beta}$, and etc.) Let z be a root of unity in the center of KBe_χ so that

$$Z(KBe_\chi) = K(z).$$

Let $\tilde{G}$ be the finite subgroup of the group of units of KGe_χ generated by z and G. Then it is easy to see that $KGe_\chi = K\tilde{G}e_\chi$. Furthermore, the subgroups $\tilde{F} = \langle z, F \rangle$, $\tilde{U} = \langle z \rangle$, and $\tilde{B} = \langle z, B \rangle$ all generate simple subalgebras of $K\tilde{G}e_\chi$, as do all normal subgroups of $\tilde{G}$ lying between $\tilde{U}$ and $\tilde{F}$. Because we have arranged that $K(\beta) \cong Z(K\tilde{F}e_\chi)$, we now have $K\tilde{F}e_\chi = K\tilde{B}e_\chi$. This allows us to use the proof of Theorem 4 (without the assumption that every normal abelian subgroup of $\tilde{G}$ is cyclic in the special case), and conclude that there is an irreducible character of $\tilde{H} = \tilde{U}\tilde{P}$ that has the same Schur index as that of $K\tilde{G}e_\chi$, for some Sylow p-subgroup $\tilde{P}$ of $\tilde{G}$. Thus we have constructively reduced the problem of determining the Schur index of KGe_χ to determining the Schur index of a simple component of the group algebra of a p-quasi-elementary group. Calculations necessary for computing the index of such an algebra have been described in [Y], [L], and [Her].

This completes the outline of the algorithm for computing Schur indices of simple components of a group algebra of a nilpotent-by-abelian group over an algebraic number field. We conclude this section with the following example that indicates the limitations of this process for general solvable groups.

For example, let E be an extraspecial group of order 27 and exponent 3. The outer automorphisms of E that fix the center of E form a group isomorphic to $SL(2,3)$. Let G be the semi-direct product $E \rtimes SL(2,3)$ in which $SL(2,3)$ acts as these outer automorphisms. The irreducible characters of G which lie over a fixed faithful irreducible character φ of E are in one-to-one correspondence with characters of $SL(2,3)$. These characters are all faithful. Simple components of $\mathbb{Q}(\zeta_3)G$ corresponding to these characters do not reduce using Clifford theory, in fact, $E = F$ and $G = C$ in the above notation. There are three characters of G that extend φ, corresponding to the linear characters of $SL(2,3)$. For these characters χ_i, $i = 1,2,3$, a reduction using Theorem 4 is possible, and it shows that the 2- and 3-parts of $m_{\mathbb{Q}(\zeta_3)}(\chi_i)$ are Schur indices of simple components of rational group algebras over the groups $C_3 \rtimes Q_8$ and $E \rtimes C_3$, respectively. Since the quaternion component of $C_3 \rtimes Q_8$ is not the one in question, and $E \rtimes C_3$ is a 3-group, we must have $m_{\mathbb{Q}}(\chi_i) = 1$ for $i = 1,2,3$. On the other hand, for the three characters of G lying over φ that satisfy $(\chi_4)_F = 2\varphi$, $(\chi_5)_F = 2\varphi$, and $(\chi_6)_F = 3\varphi$, the reduction fails because the hypothesis is not satisfied. Of course the group H can be computed, but we cannot guarantee the existence of a $\xi \in \mathrm{Irr}(H)$ that will have the right Schur index.

REFERENCES

[C] G.-Martin Cram, *On the field of character values of finite solvable groups*, Arch. Math. **51**(1988), 294–296.

[CR] Charles W. Curtis and Irving Reiner, *Methods of Representation Theory: With Applications to Finite Groups and Orders*, Wiley-Interscience, New York, 1981.

[F] Jean-Marc Fontaine, *Sur la decomposition des algebres de groups*, Ann. Scient. Ec. Norm. Sup. (4)**4**(1971), 121–180.

[G] P. X. Gallagher, *Group characters and normal Hall subgroups*, Nagoya Math J. **21**(1962), 223–230.

[Her] Allen Herman, *Metabelian groups and the Brauer-Witt theorem*, Comm. Alg. (11)**23**(1995), 4073–4086.

[Hup] Bertram Huppert, *Enliche Gruppen I*, Springer-Verlag, Berlin, 1967.

[I] I. M. Isaacs, *The Character Theory of Finite Groups*, Academic Press, New York, 1976.

[I2] ———, *Extensions of group representations over arbitrary fields*, J. Algebra **68**(1981), 54–74.

[J] G. J. Janusz, *Generators for the Schur group of local and global number fields*, Pacific J. Math. **56**(1975), 525–546.

[J2] ———, *Some remarks on Clifford's Theorem and the Schur index*, Pacific J. Math. **32**(1970), 119–125.

[L] Falko Lorenz, *Über die Berechnung der Schurshen indizes von Charakteren endlicher Gruppen*, J. Number Theory **3**(1971), 60–103.

[MW] Olaf Manz and Thomas R. Wolf, *Representations of Solvable Groups*, London Mathematical Society Lecture Note Series **185**, Cambridge University Press, Cambridge, 1993.

[P] D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley and Sons, New York, 1977.

[Roq] P. Roquette, *Arithmetische Untersuchung des Charakterringes einer endlichen Gruppen*, J. Reine Angew. Math. **190**(1952), 148–168.

[Rit] Jürgen Ritter, *Über eine Erweiterung der Witschen Reduktion bei der Bestimmung der Schurindizes von endlichen Gruppen über lokalen Körpen*, Arch. Math. **29**(1977), 78–91.

[Sch] Peter Schmidt, *Schur indices and Schur groups*, J. Algebra **169**(1994), 226–247.

[Sc] W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[Sh] M. Shirvani, *The structure of simple rings generated by finite metabelian groups*, J. Algebra **169**(1994), 686–712.

[W] E. Witt, *Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zahlkörper*, J. Reine Angew. Math. **190**(1952), 231–245.

[Y] T. Yamada, *The Schur Subgroup of the Brauer Group*, Lecture Notes in Mathematics **397**, Springer-Verlag, Berlin, 1974.

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INDUCTION AND RESTRICTION OF π-PARTIAL CHARACTERS AND THEIR LIFTS

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ABSTRACT. Let G be a finite π-separable group, where π is a set of primes. The π-partial characters of G are the restrictions of the ordinary characters to the set of π-elements of G. Such an object is said to be irreducible if it is not the sum of two nonzero partial characters and the set of irreducible π-partial characters of G is denoted $\mathrm{I}_\pi(G)$. (If p is a prime and $\pi = p'$, then $\mathrm{I}_\pi(G)$ is exactly the set of irreducible Brauer characters at p.)

From their definition, it is obvious that each partial character $\varphi \in \mathrm{I}_\pi(G)$ can be "lifted" to an ordinary character $\chi \in \mathrm{Irr}(G)$. (This means that φ is the restriction of χ to the π-elements of G.) In fact, there is a known set of canonical lifts $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ for the irreducible π-partial characters. In this paper, it is proved that if $2 \notin \pi$, then there is an alternative set of canonical lifts (denoted $\mathrm{D}_\pi(G)$) that behaves better with respect to character induction.

An application of this theory to M-groups is presented. If G is an M-group and $S \subseteq G$ is a subnormal subgroup, consider a primitive character $\theta \in \mathrm{Irr}(S)$. It was known previously that if $|G : S|$ is odd, then θ must be linear. It is proved here without restriction on the index of S that $\theta(1)$ is a power of 2.

1. **Introduction.**

This paper is intended to serve a double purpose. We obtain a few new results in the π-character theory of π-separable groups and we use some of these to study the character theory of subnormal subgroups of M-groups.

Consider M-groups first. Suppose that $S \lhd\lhd G$, where G is an M-group, and suppose $\gamma \in \mathrm{Irr}(S)$ is primitive. In [3], we showed that if S has odd index, then γ must be linear and it is a consequence of our work here that γ must also be linear in the case where S has odd order. In fact, we have more.

THEOREM A. *Let $S \lhd\lhd G$, where G is an M-group. If $\gamma \in$ Irr(S) is primitive, then $\gamma(1)$ is a power of 2.*

We mention that the case of Theorem A where $S \lhd G$ is the main theorem of [1], but the result for subnormal subgroups does not follow from this. Our actual result here is an even stronger necessary condition for a group S to be subnormally embedded in an M-group, but unfortunately, our condition does not imply that the S must be an M-group, even when it is known to have odd order.

We turn now to π-character theory, with a brief review of definitions and basic results. (The reader may wish to consult the expository papers [8], [5] and [7] for more information and for proofs of some of the key theorems.)

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Fix a set π of prime numbers and let G be a π-separable group. The *π-partial characters* of G are the restrictions of the ordinary characters to the set G^0 of π-elements of G. (We write χ^0 to denote the restriction of $\chi \in \mathrm{Char}(G)$ to G^0.) A π-partial character is *irreducible* if it cannot be written as a sum of two nonzero π-partial characters and we write $\mathrm{I}_\pi(G)$ to denote the set of all irreducible π-partial characters of G. If $\chi \in \mathrm{Char}(G)$, it is immediate that we can write $\chi^0 = \sum a_\varphi \varphi$, where φ runs over $\mathrm{I}_\pi(G)$ and the coefficients a_φ are nonnegative integers. (In fact, these coefficients are uniquely determined by χ because $\mathrm{I}_\pi(G)$ is actually a basis for the complex vector space of class functions defined on G^0.) Given $\varphi \in \mathrm{I}_\pi(G)$, we can certainly write $\varphi = \chi^0$ for some character χ of G, and in this situation it is clear that χ must lie in $\mathrm{Irr}(G)$. In other words, each irreducible π-partial character of G can be *lifted* to an ordinary irreducible character.

It is possible to 'predetermine' a complete set of lifts for the irreducible π-partial characters of a π-separable group G. There is, in other words, a canonically defined subset $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ such that the map $\chi \longmapsto \chi^0$ is a bijection from $\mathrm{B}_\pi(G)$ onto $\mathrm{I}_\pi(G)$. (The definition and principal properties of the set $\mathrm{B}_\pi(G)$ appear in [2] and an exposition of this material can be found in [7].)

The function $\mathrm{B}_\pi(\ )$, which assigns to each π-separable group G a particular set of irreducible characters of G, behaves well with respect to normal subgroups: if $\chi \in \mathrm{B}_\pi(G)$ and $N \lhd G$, then all irreducible constituents of the restriction χ_N lie in $\mathrm{B}_\pi(N)$. (As is explained in [8], this fact about the set-valued function $\mathrm{B}_\pi(\ )$ is crucial in establishing the analog of Clifford theory for the irreducible π-partial characters.)

There are some respects, however, in which the behavior of the function $\mathrm{B}_\pi(\ )$ is not under good control. Suppose, for example, that H is an arbitrary subgroup of G. If $\chi \in \mathrm{B}_\pi(G)$, then even if we know that χ_H is irreducible, we cannot always conclude that χ_H lies in $\mathrm{B}_\pi(H)$. Similarly, if we start with $\psi \in \mathrm{B}_\pi(H)$, where ψ^G is known to be irreducible, it does not follow that ψ^G lies in $\mathrm{B}_\pi(G)$.

In the case where G has odd order, it is an easy consequence of Lemma 3.1 of [6] that the restriction and induction conclusions of the previous paragraph actually are valid. In fact, an even stronger induction result holds: given that $|G|$ is odd and that $\psi^G = \chi \in \mathrm{Irr}(G)$, then $\psi \in \mathrm{B}_\pi(H)$ if and only if $\chi \in \mathrm{B}_\pi(G)$. (The additional information here, of course, is the 'if' assertion, which appears as Theorem 8.6 of [8].)

For groups of even order, one can use Lemma 3.3 of [6] to prove weak forms of the restriction and induction theorems for $\mathrm{B}_\pi(\ )$, but only when $2 \in \pi$. In the case where $2 \notin \pi$, on the other hand, strong replacements for these theorems are available, provided that we are willing to make certain changes. Specifically, we must replace $\mathrm{B}_\pi(\ )$ by a suitable alternative function $\mathrm{D}_\pi(\ )$ that also assigns a certain subset of $\mathrm{Irr}(G)$ to each π-separable group G and we must work with a twisted form of character induction called π-induction.

Definitions and properties of π-induction and of the function $\mathrm{D}_\pi(\ )$ can be found in [4], and we review some of this material in Section 2, following this introduction. We prove here that $\mathrm{D}_\pi(G)$ (like $\mathrm{B}_\pi(G)$) is a set of lifts for the irreducible π-partial characters of G. In general, $\mathrm{D}_\pi(G)$ can be different from $\mathrm{B}_\pi(G)$ and we stress that $\mathrm{D}_\pi(G)$, unlike $\mathrm{B}_\pi(G)$, is defined only when $2 \notin \pi$. (We mention that the definition of $\mathrm{D}_\pi(G)$ was suggested by unpublished work of E. C. Dade.)

THEOREM B. *Let G be π-separable, where $2 \notin \pi$. Then the map $\chi \mapsto \chi^0$ defines a bijection from* $\mathrm{D}_\pi(G)$ *onto* $\mathrm{I}_\pi(G)$.

Using Theorem B, it is easy to deduce the following.

COROLLARY C. *If G has odd order and $2 \notin \pi$, then* $\mathrm{D}_\pi(G) = \mathrm{B}_\pi(G)$.

By Theorem 7.10 of [4], the function $\mathrm{D}_\pi(\ )$ (like $\mathrm{B}_\pi(\ )$) respects restriction to normal subgroups, but unlike $\mathrm{B}_\pi(\ )$, it is also reasonably well-behaved upon restriction to non-normal subgroups.

THEOREM D. *Let $H \subseteq G$, where G is π-separable and $2 \notin \pi$. If $\chi \in$* $\mathrm{D}_\pi(G)$ *and $\psi = \chi_H$ is irreducible, then $\psi \in$* $\mathrm{D}_\pi(H)$.

In view of Corollary C, we see that for odd-order groups, Theorem D implies the restriction theorem for $\mathrm{B}_\pi(\ )$, to which we referred earlier.

The induction theorem for the function $\mathrm{D}_\pi(\ )$ is known. It was first proved in unpublished work of Dade and it appears as Theorem 7.3 of [4]. For comparison with Theorem D and because together with Theorem B, we use it to prove our M-group theorem, we state this result below as Theorem E.

We must first give at least a cursory description of π-induction, however. (More detail can be found in Section 2, and still more appears in [4].) If $2 \notin \pi$ and G is π-separable, then π-induction from a subgroup $H \subseteq G$ is a certain map $\theta \mapsto \theta^{\pi G}$ from Char(H) to Char(G). Like the ordinary induced character θ^G, the π-induced character $\theta^{\pi G}$ has degree equal to $|G : H|\,\theta(1)$. More generally, $\theta^{\pi G}(x) = \theta^G(x)$ for all odd-order elements $x \in G$. (For odd-order groups, therefore, π-induction is the same as ordinary induction, and so by Corollary C, the induction theorem for $\mathrm{B}_\pi(\ )$ in odd-order groups (to which we referred earlier) is a consequence of Theorem E.)

THEOREM E. *Let $H \subseteq G$, where G is π-separable and $2 \notin \pi$. Suppose $\psi \in$* Irr(H) *and that $\chi = \psi^{\pi G}$ is irreducible. Then $\chi \in$* $\mathrm{D}_\pi(G)$ *iff $\psi \in$* $\mathrm{D}_\pi(H)$.

We mention that the 'if' part of Theorem E is immediate from the definition of the function $\mathrm{D}_\pi(\ )$ and the properties of π-induction; it is only the 'only if' part that requires work.

Since we are assuming that $2 \notin \pi$ and we know that θ^G and $\theta^{\pi G}$ agree on elements of odd order, we see that these characters agree on the set G^0 of π-elements of G. It follows that $(\theta^{\pi G})^0 = (\theta^G)^0$ for all characters θ of H, and so from the point of view of π-partial characters, π-induction is exactly the same as ordinary induction. It is induction of π-partial characters that we consider next.

Recall from [8] that a character $\chi \in \mathrm{Irr}(G)$ is *supermonomial* if whenever $\chi = \psi^G$, where ψ is a character of a subgroup of G, then ψ is monomial. Analogously, a partial character $\varphi \in \mathrm{I}_\pi(G)$ is *monomial* if it is induced from a linear π-partial character of a subgroup and φ is *supermonomial* if every π-partial character that induces it is monomial. Our principal supermonomiality result, used to prove Theorem A, is the following.

THEOREM F. *Let G be p-solvable, where p is an odd prime. Choose $\varphi \in \mathrm{I}_p(G)$ and let χ be the unique lift of φ in $\mathrm{D}_p(G)$. The following are then equivalent.*

- *(i) φ is monomial.*
- *(ii) χ is monomial.*
- *(iii) Both φ and χ are supermonomial.*

The fact that monomial characters in $\mathrm{D}_p(G)$ must be supermonomial is a generalization of Theorem 10.2 of [8], which is the odd-order case of this result. (In [8], the theorem is stated for $\mathrm{B}_p(G)$, but we know that $\mathrm{B}_p(G) = \mathrm{D}_p(G)$ when $|G|$ is odd.)

Theorem A is not the only result about primitive characters of subgroups of M-groups that can be proved using techniques of this paper. Instead of discussing other theorems of this type here, we refer the reader to [9].

2. **Sign characters and twisted induction.** In this section we recall some definitions and facts from [4], and in particular, we review the definition of the subset $\mathrm{D}_\pi(G) \subseteq \mathrm{Irr}(G)$. The key to this is the *π-standard sign character* $\delta_{(G:H)}$, defined for each subgroup $H \subseteq G$, where G is π-separable and $2 \notin \pi$. This is a linear character of H that has values ± 1; it is determined by the action of H on the π-factors of an H-composition series for G. (We refer the reader to Section 2 of [4] for the full definition of $\delta_{(G:H)}$.) Among the properties of the π-standard sign character are the following, which are sufficient to determine it uniquely.

LEMMA 2.1. *Given $H \subseteq G$, where G is π-separable and $2 \notin \pi$, we have the following.*

- *(a) If $|G : H|$ is a π'-number, then $\delta_{(G:H)} = 1_H$.*
- *(b) If $|G : H|$ is a π-number and H is a maximal subgroup, then $\delta_{(G:H)}$ is the permutation sign character of the action of H on the right cosets of H in G.*
- *(c) If $H \subseteq K \subseteq G$, then $\delta_{(G:H)} = (\delta_{(G:K)})_H \delta_{(K:H)}$.*

PROOF. Parts (a) and (c) are respectively Theorem 2.5(c) and Theorem 2.5(b) of [4], while (b) is immediate from Corollary 2.9(a) of that paper and the fact that the sign character of an action is exactly the determinant of the associated permutation representation. ∎

Continuing to assume that G is π-separable, with $2 \notin \pi$, we recall from [4] that if $\theta \in \mathrm{Char}(H)$, then the *$\pi$-induced* character $\theta^{\pi G}$ of G is the character $(\delta\theta)^G$, where $\delta = \delta_{(G:H)}$. Since $\delta_{(G:H)}$ is trivial if $|G : H|$ is a π'-number, we see that π-induction from subgroups of π'-index agrees with ordinary induction. Also, for arbitrary subgroups $H \subseteq G$, if $x \in G$ has odd order, then $\theta^{\pi G}(x) = \theta^G(x)$. (This is because $\delta(y) = 1$ for every element y of H that is conjugate in G to x.)

In general, π-induction enjoys many of the familiar properties of ordinary induction. It is transitive, for example. In other words, if $H \subseteq K \subseteq G$ and $\theta \in \mathrm{Char}(H)$, then $(\theta^{\pi K})^{\pi G} = \theta^{\pi G}$. (This is an immediate consequence of Lemma 2.1 (c).) Somewhat more interesting is the situation where $G = XY$ for subgroups X and Y. If $\theta \in \mathrm{Char}(Y)$, it is a well-known consequence of Mackey's theorem that $(\theta^G)_X = (\theta_{X \cap Y})^X$. The analog of this formula for π-induction would be immediate if we knew that $(\delta_{(G:Y)})_{X\cap Y} = \delta_{(X:X\cap Y)}$, but in [4], this was proved only under the additional assumption that one of X or Y is the product of $X \cap Y$ with a normal subgroup of G. In fact, it is not very difficult to prove the general case. (In order to give a self-contained proof here, we do not assume the part of this result that appears in [4].) We require the following lemma, which, except for its dependence on the odd-order theorem, is entirely routine.

LEMMA 2.2. *Let G be π-separable, where $2 \notin \pi$, and suppose $\mathbf{O}_{\pi'}(G) = 1$. Then all core-free maximal subgroups of G are conjugate.*

PROOF. Let N be a minimal normal subgroup of G. Then N is a π-group, and thus is of odd order and hence is an abelian p-group for some prime $p \in \pi$. If X is a core-free maximal subgroup of G, then $N \not\subseteq X$ and thus $NX = G$ and $N \cap X = 1$. It follows that $\mathbf{C}_G(N) \cap X \lhd G$, and we conclude that $\mathbf{C}_G(N) = N$.

If $N = G$ then $X = 1$ and there is nothing to prove. We can thus assume that $N < G$ and we let K/N be a chief factor of G. We claim that K/N is a p'-group. This is certainly true if K/N is a π'-group, and so we can assume it is a π-group. It thus has odd order, and hence it is an abelian q-group for some prime q. But $q \neq p$; otherwise, K is a p-group and $N \subseteq \mathbf{Z}(K)$, which contradicts the fact that N is self-centralizing.

We see now that $X \cap K$ is a p-complement in K and that $X = \mathbf{N}_G(X \cap K)$. Since all p-complements in K are conjugate, their normalizers in G are also conjugate, and the result follows. ■

LEMMA 2.3. *Let $X, Y \subseteq G$, where G is π-separable and $2 \notin \pi$. Assume that $XY = G$ and write $D = X \cap Y$. Then*

(a) $(\delta_{(G:Y)})_D = \delta_{(X:D)}$.

(b) $(\delta_{(G:X)})_D = \delta_{(Y:D)}$.

(c) If $\theta \in \mathrm{Char}(Y)$, *then* $(\theta^{\pi G})_X = (\theta_D)^{\pi X}$.

PROOF. We observe first that (a) and (b) imply each other for each choice of subgroups X and Y with $XY = G$. To see this, we argue that since $\delta_{(Y:D)}$ is nonvanishing, the assertion of (a) is equivalent to the equation $\delta_{(X:D)}\delta_{(Y:D)} = (\delta_{(G:Y)})_D\delta_{(Y:D)}$. Since the right side of this equation is equal to $\delta_{(G:D)}$ by Lemma 2.1(a), it follows that (a) is true iff $\delta_{(G:D)} = \delta_{(X:D)}\delta_{(Y:D)}$, which is symmetric in X and Y. It follows that (b) is also equivalent to this equation, and hence (a) and (b) are equivalent to each other.

We now prove parts (a) and (b) by induction on the index $|G : D|$. If $X = G$, then $D = Y$ and (a) is trivially true, and it follows that (b) holds too. Similarly, (a) and (b) hold if $Y = G$, and thus we may thus assume that X and Y are each proper in G.

Suppose that $X < H < G$ for some subgroup H. Observe that $HY = G$ and that $XE = H$, where we have written $E = H \cap Y$. Since $H > X$, we see that $E > D$ and thus $|G : E| < |G : D|$. Also $|H : D| < |G : D|$ since $H < G$, and hence two applications of the inductive hypothesis yield that $\delta_{(X:D)} = (\delta_{(H:E)})_D = \left((\delta_{(G:Y)})_E\right)_D$. Thus (a) holds, and so (b) holds too, and similarly, both (a) and (b) hold if Y is not maximal in G.

We can now assume that each of X and Y is a maximal subgroup of G. In particular, since G is π-separable, each of the indices $|G : X|$ and $|G : Y|$ is either a π-number or a π'-number. Suppose that one of these indices (say $|G : Y|$) is a π'-number. Then also $|X : D|$ is a π'-number and hence $\delta_{(G:Y)} = 1_Y$ and $\delta_{(X:D)} = 1_D$ and assertion (a) follows. Since (a) and (b) both hold if either of the two indices $|G : X|$ or $|G : Y|$ is a π'-number, we can assume that both indices are π-numbers and hence $|G : D|$ is a π-number.

Now let $M = \mathrm{core}_G(D)$ and suppose that $M < \mathrm{core}_G(Y)$. Writing $N = \mathrm{core}_G(Y)$, we see that $N \not\subseteq X$ and thus $NX = G$ since X is maximal in G. In this situation, $Y = ND$ and we claim that D is a maximal subgroup of X. To see this, suppose $D < U < X$. Then $YU = NU$ is a subgroup and $Y < YU < G$ since $YU \cap X = U$, and this contradicts the maximality of Y in G.

Since D is maximal and of π-index in X, we know from Lemma 2.1(b) that $\delta_{(X:D)}$ is the permutation sign character of the action of D on the right cosets of D in X. This action, however, is permutation isomorphic to the action of D on the right cosets of Y in G. (Intersection of a coset of Y with X gives the desired isomorphism.) Since $\delta_{(G:Y)}$ is the permutation sign character of the action of Y on the right cosets of Y in G, we see that $(\delta_{(G:Y)})_D$ is the permutation sign character of the action of D on these cosets. In this situation, we have $(\delta_{(G:Y)})_D = \delta_{(X:D)}$ and assertions (a) and (b) follow.

We can now assume that M is the core of Y in G and similarly, it is the core of X. Working in G/M, therefore, we see that X/M and Y/M are core-free maximal subgroups. They have π-index, and so they contain $\mathbf{O}_{\pi'}(G/M)$, which is therefore trivial. It follows from Lemma 2.2 that X/M and Y/M are conjugate, and thus X and Y are conjugate in G. This is impossible, however, since $XY = G$, and this contradiction completes the proof of (a) and (b).

The proof of (c) is now immediate because

$$(\theta^{\pi G})_X = \big((\delta_{(G:Y)}\theta)^G\big)_X = \big((\delta_{(G:Y)}\theta)_D\big)^X = (\delta_{(X:D)}\theta_D)^X = (\theta_D)^{\pi X}$$

where the penultimate equality follows from (a). ∎

We turn now to the definition of the set $\mathrm{D}_\pi(G)$. If G is π-separable, we recall (for comparison) that $\mathrm{B}_\pi(G)$ is a superset of the set $X_\pi(G)$ of π-special characters. In fact, the π-special characters of G are exactly those members of $\mathrm{B}_\pi(G)$ that have π-degree and every member of $\mathrm{B}_\pi(G)$ is induced from an appropriate π-special character of a subgroup of G. (See [7] for an exposition of this.) Given that $2 \notin \pi$, the set $\mathrm{D}_\pi(G)$ is also a superset of $X_\pi(G)$. By the definition given in [4], it consists of those irreducible characters of G that can be obtained via π-induction from π-special characters of subgroups of G. In other words, $\chi \in \mathrm{Irr}(G)$ lies in $\mathrm{D}_\pi(G)$ precisely when there exists a π-special character θ of some subgroup $U \subseteq G$ such that $\theta^{\pi G} = \chi$.

Note that by the transitivity of π-induction, it is clear that if $H \subseteq G$ and $\psi \in \mathrm{D}_\pi(H)$ with $\psi^{\pi G} = \chi \in \mathrm{Irr}(G)$, then $\chi \in \mathrm{D}_\pi(G)$. This proves the 'if' part of Theorem E. (Theorem E is fully proved in [4], and we say no more about it here.)

We can now prove Theorem D.

PROOF OF THEOREM D. We are given $\chi \in \mathrm{D}_\pi(G)$ such that $\psi = \chi_H$ is irreducible and our task is to show that $\psi \in \mathrm{D}_\pi(H)$. By definition of $\mathrm{D}_\pi(G)$, we can write $\chi = \theta^{\pi G}$ for some π-special character θ of some subgroup $U \subseteq G$. Since χ is induced from a character of U and it restricts irreducibly to H, it follows that $UH = G$. By Lemma 2.3(c), we have $\psi = \chi_H = (\theta_{U\cap H})^{\pi H}$, and we know that this is irreducible. It follows that $\theta_{U\cap H}$ is irreducible, and thus it is π-special by Theorem A of [4]. By definition, therefore, ψ lies in $\mathrm{D}_\pi(H)$, as required. ∎

By Corollary 7.2 of [4], the π-special characters of G are exactly those members of $\mathrm{D}_\pi(G)$ that have π-degree. For the remaining members of $\mathrm{D}_\pi(G)$ we shall need the following technical result, which is part of Lemma 7.7 of [4]. We quote it here without proof.

LEMMA 2.4. *Let G be π-separable, where $2 \notin \pi$, and suppose that $\chi \in \mathrm{D}_\pi(G)$. If $\chi(1)$ is not a π-number, then there exists $N \lhd G$ such that χ_N has π-special irreducible constituents that are not invariant in G.* ∎

3. **Lifting π-partial characters.**

In this section we prove Theorem B and its corollary.

PROOF OF THEOREM B. Given $\varphi \in \mathrm{I}_\pi(G)$, we must produce a character $\chi \in \mathrm{D}_\pi(G)$ such that $\chi^0 = \varphi$. There exists $\psi \in \mathrm{B}_\pi(G)$ with $\psi^0 = \varphi$ and by definition of the set $\mathrm{B}_\pi(G)$ in [2] (or see the exposition in [7]), there exists a π-special character γ of some subgroup $W \subseteq G$ such that $\gamma^G = \psi$. (As explained in [7], we can take the pair (W, γ) to be a 'nucleus' for ψ.)

Now let $\chi = \gamma^{\pi G} = (\delta\gamma)^G$, where $\delta = \delta_{(G:W)}$ is a certain linear character of W such that $\delta^2 = 1_W$. If $x \in G$ is a π-element, then since $2 \notin \pi$, we know that x has odd order, and it follows that

$$\chi(x) = \gamma^{\pi G}(x) = \gamma^G(x) = \psi(x) = \varphi(x)$$

and thus $\chi^0 = \varphi$. Since φ is an irreducible π-partial character, it follows that $\chi \in \mathrm{Irr}(G)$ and hence by definition, $\chi \in \mathrm{D}_\pi(G)$.

To complete the proof, we show by induction on $|G|$ that the map $\chi \mapsto \chi^0$ is an injection from $\mathrm{D}_\pi(G)$ into $\mathrm{I}_\pi(G)$. Let $\chi \in \mathrm{D}_\pi(G)$ and write $\varphi = \chi^0$. We must prove that φ is irreducible and that if also $\xi \in \mathrm{D}_\pi(G)$ with $\xi^0 = \varphi$, then $\xi = \chi$. We suppose first that $\chi(1)$ is a π-number. Then χ is π-special, and hence $\chi \in \mathrm{B}_\pi(G)$. Also, since $\xi^0 = \varphi$, we have $\xi(1) = \chi(1)$, and thus ξ is also π-special and lies in $\mathrm{B}_\pi(G)$. But we know that restriction to π-elements defines a bijection from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$, and we deduce that $\chi = \xi$ and that φ is irreducible, as desired.

We can now assume that $\chi(1)$ is not a π-number. By Lemma 2.4, there exists a subgroup $N \lhd G$ such that an irreducible constituent θ of χ_N is π-special and is not invariant in G. Let $T < G$ be the stabilizer of θ in G and observe that T is also the stabilizer of θ^0, which is an irreducible π-partial character of N. (This is because $\theta \in \mathrm{B}_\pi(N)$ and restriction to π-elements defines a bijection from $\mathrm{B}_\pi(N)$ to $\mathrm{I}_\pi(N)$.)

We have $\chi = \alpha^G$ for some uniquely determined Clifford correspondent $\alpha \in \mathrm{Irr}(T|\theta)$. If we write $\delta = \delta_{(G:T)}$ and $\psi = \delta\alpha$, we see that $\psi^{\pi G} = \alpha^G = \chi$, and since $\chi \in \mathrm{D}_\pi(G)$, we deduce from Theorem E that $\psi \in \mathrm{D}_\pi(T)$. Clearly, $(\psi^0)^G = \varphi$ and ψ^0 lies over $\theta^0 \in \mathrm{I}_\pi(N)$.

By the inductive hypothesis, we know that $\psi^0 \in \mathrm{I}_\pi(T)$, and since T is the stabilizer of θ^0 in G, the Clifford correspondence for π-partial characters guarantees that $\varphi = (\psi^0)^G$ is irreducible, as desired. (See Theorem 5.3 of [8].)

Given $\xi \in \mathrm{D}_\pi(G)$ with $\xi^0 = \varphi$, we know that θ^0 is a constituent of $\varphi_N = (\xi_N)^0$, and so θ^0 is a constituent of μ^0, where μ is some (ordinary) irreducible constituent of ξ_N. But the irreducible constituents of ξ_N all lie in $\mathrm{D}_\pi(N)$. (As we mentioned earlier, this is Theorem 7.10 of [4].) However, $N \subseteq T < G$ and the inductive hypothesis guarantees that restriction to π-elements defines a bijection from $\mathrm{D}_\pi(N)$ to $\mathrm{I}_\pi(N)$. Since both θ and μ lie in $\mathrm{D}_\pi(N)$ and θ^0 is known to be a constituent of μ^0, it follows that $\theta = \mu$, and thus θ lies under ξ and we let $\beta \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of ξ with respect to θ. Thus $\beta^G = \xi$ and we have $\eta^{\pi G} = \xi$, where $\eta = \delta\beta$. Reasoning as in the previous paragraph, we deduce that $\eta \in \mathrm{D}_\pi(T)$ and that η^0 is the Clifford correspondent of φ with respect to θ^0. It follows that $\eta^0 = \psi^0$, and thus $\eta = \psi$ by the inductive hypothesis applied to T. Thus $\xi = \eta^{\pi G} = \psi^{\pi G} = \chi$, and the proof is complete. ∎

PROOF OF COROLLARY C. Given that $|G|$ is odd, we claim that $\mathrm{B}_\pi(G) \subseteq \mathrm{D}_\pi(G)$. If $\chi \in \mathrm{B}_\pi(G)$, we know that $\chi = \gamma^G$, where γ is a π-special character of some subgroup $W \subseteq G$. But $\delta_{(G:W)} = 1_W$ since $|W|$ is odd; thus $\chi = \gamma^{\pi G}$, and hence $\chi \in \mathrm{D}_\pi(G)$. Thus $\mathrm{B}_\pi(G) \subseteq \mathrm{D}_\pi(G)$ as claimed. But by Theorem B, we have $|\mathrm{D}_\pi(G)| = |\mathrm{I}_\pi(G)| = |\mathrm{B}_\pi(G)|$, and hence $\mathrm{B}_\pi(G) = \mathrm{D}_\pi(G)$, as desired. ∎

4. **Supermonomial partial characters.**

In this section, we prove Theorem F. The principal nontrivial fact on which we rely is that for p-solvable groups with $p \neq 2$, monomial irreducible characters of p-power degree are necessarily supermonomial. This result, which is discussed in [8] and proved there as Theorem 10.1, is a relatively easy consequence of a deep result of E. C. Dade. The argument we use here to obtain Theorem F from Theorem 10.1 is really the same as the proof of Theorem 10.2 in [8], which is essentially the odd-order case of Theorem F.

We need the following easy consequence of Theorem B.

LEMMA 4.1. *Let $H \subseteq G$, where G is π-separable and $2 \notin \pi$. Let $\chi \in \mathrm{D}_\pi(G)$ and suppose that $\chi^0 = \alpha^G$, where $\alpha \in \mathrm{I}_\pi(H)$. Then $\chi = \eta^G$ for some character $\eta \in \mathrm{Irr}(H)$ such that $\eta^0 = \alpha$.*

PROOF. To find η, first choose $\psi \in \mathrm{D}_\pi(H)$ with $\psi^0 = \alpha$. (This is possible by Theorem B). Now $(\psi^{\pi G})^0 = \alpha^G = \chi^0$ is irreducible, and thus $\psi^{\pi G}$ lies in $\mathrm{D}_\pi(G)$ by the easy part of Theorem E. Since also $\chi \in \mathrm{D}_\pi(G)$ and the map $\xi \longmapsto \xi^0$ is injective on $\mathrm{D}_\pi(G)$, we deduce that $\psi^{\pi G} = \chi$. We can now take $\eta = \delta_{(G:H)}\psi$ to complete the proof. ∎

PROOF OF THEOREM F. We are given $\varphi \in \mathrm{I}_p(G)$, where G is p-solvable and $p \neq 2$. It is clear that if any lift of φ in $\mathrm{Irr}(G)$ is monomial, then φ must be monomial. We may assume, therefore, that φ is monomial, and our task is to show that φ is supermonomial and that its lift in $\mathrm{D}_p(G)$ is also supermonomial.

Write $\varphi = \alpha^G$, where α is a linear p-partial character of some subgroup $A \subseteq G$, and assume that also $\varphi = \beta^G$, where $\beta \in \mathrm{I}_p(B)$ for some subgroup $B \subseteq G$. We must show that β is monomial and we consider first, the case where $\varphi(1)$ is a power of p.

We have $\chi \in \mathrm{D}_p(G)$ with $\chi^0 = \varphi$. Since $\alpha^G = \varphi = \beta^G$, it follows by Lemma 4.1 that we can find $\psi \in \mathrm{Irr}(A)$ and $\eta \in \mathrm{Irr}(B)$ such that $\psi^G = \chi = \eta^G$, where $\psi^0 = \alpha$ and $\eta^0 = \beta$. In particular, $\psi(1) = \alpha(1) = 1$, and hence χ is a monomial character. Since χ has p-power degree, it follows by Theorem 10.1 of [8] that it is supermonomial, and hence η is monomial. We deduce that $\beta = \eta^0$ is monomial, as required.

In the general case, where $\varphi(1)$ may not be a p-power, we can assume that β is primitive. By Corollary 5.5 of [8], we know that $\beta(1)$ is a power of p and, of course $\alpha(1) = 1$ is also a power of p. In other words, in the language of Section 6 of [8], each of the pairs (A, α) and (B, β) is a p-inducing pair belonging to φ, and thus each of these pairs is a node of the graph $\mathcal{G}(\varphi)$. By Theorem 6.2 of [8], therefore, if we replace the pair (A, α) by a suitable conjugate pair, we can assume that the two nodes (A, α) and (B, β) lie in the same connected component of this graph.

Recall that by the definition of the graph, nodes (U, μ) and (V, ν) are joined in $\mathcal{G}(\varphi)$ if either $U < V$ and $\mu^V = \nu$ or $V < U$ and $\nu^U = \mu$. We claim that whenever nodes (U, μ) and (V, ν) are joined, if either of the p-partial characters μ or ν is monomial, then the other is monomial too. To see this, assume that μ is monomial. In the case where $U < V$ and $\mu^V = \nu$, it is clear that ν is monomial, and so we need consider only the situation where $V < U$ and $\nu^U = \mu$. To prove that ν is monomial in this case, it suffices to show that the monomial p-partial character μ of U is actually supermonomial. But $\mu(1)$ is a p-power (since (U, μ) was assumed to be a p-inducing pair for φ) and this is the case of the theorem we have already proved. Thus ν is monomial, as claimed.

Recall that the pairs (A, α) and (B, β) lie in the same component of $\mathcal{G}(\varphi)$. Since α is linear, it is certainly monomial, and thus by the result of the previous paragraph, β is monomial too, as required.

Finally, we must prove that χ is supermonomial, where $\chi \in \mathrm{D}_p(G)$ and $\chi^0 = \varphi$. Assuming that $\chi = \psi^G$, where $\psi \in \mathrm{Irr}(H)$ and $H \subseteq G$, we need to show that ψ is monomial. If we write $\beta = \psi^0$, we see that $\beta^G = \varphi$ and since φ is supermonomial, we know that β is monomial.

Now write $\eta = \delta_{(G:H)}\psi$, so that $\eta^{pG} = \chi$ and $\eta^0 = \psi^0 = \beta$. Then $\eta \in \mathrm{D}_p(H)$ by Theorem E, and since β is monomial, it follows by Lemma 4.1 that η is monomial. We can thus write $\eta = \lambda^H$ for some linear character $\lambda \in \mathrm{Irr}(K)$, where $K \subseteq H$, and we see that $\psi = \left((\delta_{(G:H)})_K\lambda\right)^H$ is monomial, as desired. ■

5. **One-prime M-groups.** Fix an odd prime p. We shall say that a p-solvable group G is a D_pM-*group* if every character $\chi \in \mathrm{D}_p(G)$ is monomial. Of course, an M-group is automatically a D_pM-group for every odd prime p. (Note also that every group of order not divisible by p is a D_pM-group since $\mathrm{D}_p(G)$ consists just of the principal character in this case.)

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One of the principal outstanding conjectures about M-groups is that normal (and hence also subnormal) subgroups of odd-order M-groups must also be M-groups. We prove the corresponding assertion for D_pM-groups, and from this we obtain Theorem A of the introduction.

THEOREM 5.1. *If $N \lhd\lhd G$, where G is a $\mathrm{D}_p M$-group, then N is also a $\mathrm{D}_p M$-group.*

It follows that subnormal subgroups of M-groups must be D_pM-groups for all odd primes p. This imposes a severe restriction on the collection of groups that can be subnormally embedded in an M-group, but it does not settle the normal-subgroup conjecture since there do exist odd-order groups that are D_pM-groups for all odd primes p, and yet are not M-groups.

Theorem 5.1 is an easy consequence of the following stronger result.

THEOREM 5.2. *Let $N \lhd\lhd G$, where G is p-solvable for some odd prime p. If $\chi \in \mathrm{D}_p(G)$ is monomial, then every irreducible constituent of χ_N is monomial.*

PROOF. Since all irreducible constituents of χ_S lie in $\mathrm{D}_p(S)$ for all subnormal subgroups S of G, it suffices to prove the result in the case where N is a maximal normal subgroup of G. We assume this, therefore, and we let ψ be an irreducible constituent of χ_N.

Let T be the stabilizer in G of ψ and let $\eta \in \mathrm{Irr}(T \mid \psi)$ be the Clifford correspondent of χ with respect to ψ. Then $\eta^G = \chi$, and by Theorem F, χ is supermonomial (since it is monomial). It follows that η is monomial and we can write $\eta = \lambda^G$ for some linear character λ of some subgroup of $H \subseteq T$. If we can show that $\eta_N = \psi$, it will follow that $\psi = (\lambda_{H\cap N})^N$, as required.

We know that $\eta_N = e\psi$, where e is the multiplicity of ψ as a constituent of χ_N. Our task, therefore, is to show that this multiplicity is equal to 1. If $|G : N| = p$, this is clear, and thus we can assume that $|G : N|$ is not divisible by p. Write $\varphi = \chi^0$ and $\theta = \psi^0$, and note that both φ and θ are irreducible p-partial characters. Also, the multiplicity of θ as a constituent of φ_N is 1 since p does not divide $|G : N|$. (See, for example, Corollary 5.1 of [8].) It follows that the multiplicity of ψ as a constituent of χ_N cannot exceed 1, and the proof is complete. ∎

PROOF OF THEOREM 5.1. Let $\psi \in \mathrm{D}_p(N)$. To show that ψ is monomial, it suffices by Theorem 5.2 to find a character $\chi \in \mathrm{D}_p(G)$ that lies over ψ. Now $\theta = \psi^0$ lies in $\mathrm{I}_p(N)$ and it is easy to see that there exists $\varphi \in \mathrm{I}_p(G)$ such that θ is a constituent of φ_N. (This is Lemma 4.3 of [8].) By Theorem B, choose $\chi \in \mathrm{D}_p(G)$ such that $\chi^0 = \varphi$ and observe that since N is subnormal, all irreducible constituents of χ_N lie in $\mathrm{D}_p(N)$. Since θ is a constituent of $(\chi_N)^0$, it follows by Theorem B that ψ must be a constituent of χ_N, and the proof is complete. ∎

Recall that a primitive irreducible character of a π-separable group can be factored as a product of a π-special character and a π'-special character. (This theorem was proved first in [1], and it is discussed in [7], where it appears as Theorem 2.4.) We can use this factorization of primitive characters to prove the following extension of Theorem A.

COROLLARY 5.3. *Let $N \lll G$, where G is a D_pM-group for some odd prime p. If $\psi \in \mathrm{Irr}(N)$ is primitive, then $\psi(1)$ is not divisible by p.*

PROOF. Factor $\psi = \alpha\beta$, where α is p-special and β is p'-special. Since α is p-special, it lies in $\mathrm{D}_p(N)$, and hence it is monomial by Theorem 5.1. But α is a factor of a primitive character, and thus α is primitive, and hence is linear. We conclude that $\psi(1) = \beta(1)$ is not divisible by p, as required. ■

PROOF OF THEOREM A. We are given that $N \lll G$, where G is an M-group, and thus N is a D_pM-group for all odd primes p. By Corollary 5.3, the degree of a primitive irreducible character of N can be divisible by no odd prime, and thus it is a power of 2. ■

6. **Examples and further remarks.** We mentioned that in general, the function $\mathrm{B}_\pi(\ )$ does not respect restriction to arbitrary subgroups. If $|G|$ is odd, however, and $\chi \in \mathrm{B}_\pi(G)$, then by Corollary 8.4 of [8], there is always a constituent of χ_H in $\mathrm{B}_\pi(H)$ for subgroups $H \subseteq G$. In particular, if we know that χ_H is irreducible, then it lies in $\mathrm{B}_\pi(H)$. (This is the result that Theorem D generalizes.) If $|G|$ is even, however, then it is possible to have $\chi \in \mathrm{B}_\pi(G)$ with χ_H irreducible, but with $\chi_H \notin \mathrm{B}_\pi(H)$. This can even happen if χ is π-special, as is shown in Example 8.2 of [4]. If χ is π-special and $2 \notin \pi$, however, no such example can occur. We have been unable to decide the following, however.

QUESTION 6.1. Let G be π-separable, where $2 \notin \pi$, and suppose $\chi \in \mathrm{B}_\pi(G)$. Suppose $H \subseteq G$ and that χ_H is irreducible. Must we have $\chi_H \in \mathrm{B}_\pi(H)$ in this case?

Of course, if it were always true that $\mathrm{B}_\pi(G) = \mathrm{D}_\pi(G)$ for π-separable groups G with $2 \notin \pi$, then Question 6.1 would have an affirmative answer, by Theorem D. In fact, however, $\mathrm{B}_\pi(G)$ and $\mathrm{D}_\pi(G)$ are different, in general.

EXAMPLE 6.2. Let $\pi = \{3\}$. Then there exists a π-separable group G for which $\mathrm{B}_\pi(G) \neq \mathrm{D}_\pi(G)$.

PROOF. Let G be the semidirect product of the symmetric group on four symbols, acting on an elementary abelian group V of order 3^4, where the symmetric group permutes a basis $\{v_1, v_2, v_3, v_4\}$ for V in its natural action. Let λ be the unique linear character of V such that $\lambda(v_1) = \lambda(v_2) = 1$, $\lambda(v_3) = \omega$ and $\lambda(v_4) = \overline{\omega}$, where ω is a fixed primitive cube root of unity.

The stabilizer of λ in S_4 is exactly the transposition $t = (1, 2)$, and thus V has index 2 in the stabilizer T of λ in G. We see that λ has exactly two extensions α and β to T, and these are distinguished by the values $\alpha(t) = 1$ and $\beta(t) = -1$. Write $\alpha^G = \chi$ and $\beta^G = \psi$ and observe that by the Clifford correspondence, these are distinct irreducible characters of G.

We see that $\alpha^0 = \beta^0$, and thus $\chi^0 = \psi^0 \in \mathrm{I}_\pi(G)$ by the Clifford correspondence for partial characters. Since $\chi \neq \psi$ and each is a lift of the same π-partial character, it suffices to show that $\chi \in \mathrm{B}_\pi(G)$ and $\psi \in \mathrm{D}_\pi(G)$.

First, we consider χ. It is easy to check that the pair (V, λ) is maximal among π-factored subnormal pairs in G. (See Section 4 of [7] for the relevant definitions.) Since

T is the stabilizer of this pair and $\alpha \in \mathrm{Irr}(T)$ is π-special and lies over λ, we deduce that $\chi = \alpha^G$ lies in $\mathrm{B}_\pi(G)$, as claimed.

To see that $\psi \in \mathrm{D}_\pi(G)$, it suffices to show that $\alpha^{\pi G} = \psi$, and for this purpose, we check that $\alpha\delta_{(G:T)} = \beta$. It suffices, therefore, to verify that $\delta_{(G:T)}(t) = -1$. To compute $\delta_{(G:T)}$, we choose a subgroup P such that $T \subseteq P \subseteq G$, where $|G : P| = 3$. Then $|P : T| = 4$, and hence $\delta_{(P:T)} = 1_T$ and $\delta_{(G:T)}(t) = \delta_{(G:P)}(t)$ by Lemma 2.1(a) and (c). Furthermore, since P is maximal in G, we see by Lemma 2.1(b) that $\delta_{(G:P)}$ is the permutation sign character of the action of P on the three right cosets of P in G. Since t is not in the core of P, the action of t is nontrivial, and thus it is an odd permutation. It follows that $\delta_{(G:P)}(t) = -1$, as desired. ∎

It is amusing to note that in general, the symmetric difference of the sets $\mathrm{B}_\pi(G)$ and $\mathrm{D}_\pi(G)$ consists of characters with degrees that are neither π-numbers nor π'-numbers. That a character with π-degree in either set must be in both is a consequence of the fact that such a character must be π-special. On the other hand, if $\chi \in \mathrm{B}_\pi(G)$, then we know that $\chi = \gamma^G$ for some π-special character γ of a subgroup $W \subseteq G$. If χ has π'-degree, then W must have π'-index, and thus $\delta_{(G:W)}$ is trivial and $\chi \in \mathrm{D}_\pi(G)$. Our assertion now follows (using Theorem B).

Finally, we consider the class of odd-order groups that can occur as normal (or subnormal) subgroups of M-groups. We know by Theorem 5.1 that such a group must be a D_pM-group for every odd prime p. If it were true that only M-groups had this property, this would prove the conjecture that odd-order normal subgroups of M-groups must be M-groups. Unfortunately, we have the following.

EXAMPLE 6.3. There exists an odd-order group G of that is a D_pM-group for every odd prime p, but that is not an M-group.

In fact, it is relatively easy to construct groups that are D_pM-groups for all odd primes.

LEMMA 6.4. *Suppose that $N \lhd G$, where N is a normal abelian Sylow q-subgroup of G and G/N is a D_pM-group for all odd primes p. Then G is a D_pM-group for all odd primes p.*

PROOF. Let $\chi \in \mathrm{D}_p(G)$. Then the irreducible constituents of χ_N lie in $\mathrm{D}_p(N)$, and so if $p \neq q$, these constituents are trivial and $N \subseteq \ker\chi$. In this case, we can view χ as a character of G/N, and it is easy to see that $\chi \in \mathrm{D}_p(G/N)$. (We omit the details of this routine argument.) If $p \neq q$, therefore, χ is monomial as a character of G/N and hence also as a character of G.

We can suppose now that $p = q$ so that $\chi = \theta^{qG}$ for some q-special character θ of some subgroup $H \subseteq G$. Since θ is q-special, however, it restricts irreducibly to a Sylow q-subgroup of H, which is abelian. It follows that θ is linear and thus $\chi = (\delta_{(G:H)}\theta)^G$ is monomial, as required. ∎

PROOF OF EXAMPLE 6.3. Choose any odd-order M-group H that has a subgroup K that is not an M-group. (Examples of this abound.) Next, choose an odd prime q not dividing $|H|$ and let N be an elementary abelian q-group of order $q^{|H:K|}$. Let H act on N

by permuting a basis so that K is the full stabilizer of one of the basis vectors and let G be the semi-direct product of N by H.

Since G/N is an M-group, it is certainly a D_pM-group for all odd primes p. By Lemma 6.4, therefore, G is a D_pM-group for all odd primes p, and it suffices to show that G is not an M-group.

There exists a linear character λ of N whose stabilizer in H is exactly K, and so its stabilizer in G is $T = NK$. Since λ extends to T and T/N is not an M-group, it is easy to see that T has a non-monomial irreducible character θ lying over λ. Also, $\chi = \theta^G$ is irreducible by the Clifford correspondence, and we claim that χ is not monomial.

Suppose that $\chi = \alpha^G$ for some linear character α of a subgroup X of G. We see that $|G : X| = \chi(1)$ is not divisible by q, and hence $N \subseteq X$. Since α_N lies under χ, it is G-conjugate to λ, and so we can replace the pair (X, α) by a conjugate pair and suppose that $\alpha_N = \lambda$. Then $X \subseteq T$ and $\alpha^T \in \mathrm{Irr}(T)$ lies over λ and under χ. It follows that $\alpha^T = \theta$, and this is a contradiction since θ is not monomial. ∎

We close with something only marginally related to the topic of this paper. In Theorem E of [4], it was shown that if $H \subseteq G$, where G is π-separable and $2 \notin \pi$, and if $\psi \in \mathrm{Irr}(H)$ is π-special and extends to G, then in fact, ψ has a π-special extension to G. It was asked in [4] whether or not this result might remain valid without the assumption that $2 \notin \pi$. In fact, the answer is 'yes', and even more is true, both in the case where $2 \in \pi$ and where $2 \notin \pi$.

COROLLARY 6.5. *Let $H \subseteq G$, where G is π-separable, and suppose that $\theta \in \mathrm{I}_\pi(H)$ extends to $\varphi \in \mathrm{I}_\pi(G)$.*

(a) If $2 \in \pi$, then the lift of θ in $\mathrm{B}_\pi(H)$ extends to the lift of φ in $\mathrm{B}_\pi(G)$.

(b) If $2 \notin \pi$, then the lift of θ in $\mathrm{D}_\pi(H)$ extends to the lift of φ in $\mathrm{D}_\pi(G)$.

PROOF. Suppose first that $2 \in \pi$ and let $\chi \in \mathrm{B}_\pi(G)$ with $\chi^0 = \varphi$. Write $\psi = \chi_H$ and observe that $\psi^0 = \theta$, so that ψ is irreducible. By results of [6], there is a 'magic' field automorphism that fixes χ, and so it also fixes $\psi = \chi_H$. As $(\psi)^0 = \theta$ is irreducible, Lemma 3.3 of [6] guarantees that $\psi \in \mathrm{B}_\pi(H)$, as required.

Assuming now that $2 \notin \pi$, let $\chi \in \mathrm{D}_\pi(G)$ lift φ and again write $\psi = \chi_H$. Then $\psi^0 = \theta$, and so ψ is irreducible, and it follows that $\psi \in \mathrm{D}_\pi(H)$ by Theorem D. ∎

COROLLARY 6.6. *Let H be π-separable and suppose $\theta \in \mathrm{I}_\pi(H)$. Then θ has a lift $\psi \in \mathrm{Irr}(H)$ such that whenever $H \subseteq G$, where G is π-separable, ψ extends to a character of G iff θ extends to a π-partial character of G.*

PROOF. If $2 \in \pi$, take $\psi \in \mathrm{B}_\pi(H)$ and otherwise, take $\psi \in \mathrm{D}_\pi(H)$. ∎

REFERENCES

1. I. M. Isaacs, *Primitive characters, normal subgroups and M-groups*, Math. Zeit. **177**(1981), 267–284.
2. ———, *Characters of π-separable groups*, J. Algebra **86**(1984), 98–128.
3. ———, *Characters of subnormal subgroups of M-groups*, Arch. Math. **42**(1984), 509–515.
4. ———, *Induction and restriction of π-special characters*, Canad. J. Math. **38**(1986), 576–604.
5. ———, *Partial characters of π-separable groups*, Prog. Math. **95**(1991), 273–287.

6. ———, *Characters and Hall subgroups of groups of odd order*, J. Algebra **157**(1993), 548–561.
7. ———, *The π-character theory of solvable groups*, J. Austral. Math. Soc. (Ser. A) **57**(1994), 81–102.
8. ———, *Characters and sets of primes for solvable groups*, Proc. of NATO ASI: Finite and locally finite groups (1995), 347–376.
9. M. Lewis, *Primitive characters of subgroups of M-groups*, to appear.

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GREENBERG'S THEOREM FOR QUASICONVEX SUBGROUPS OF WORD HYPERBOLIC GROUPS

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ABSTRACT. Analogues of a theorem of Greenberg about finitely generated subgroups of free groups are proved for quasiconvex subgroups of word hyperbolic groups. It is shown that a quasiconvex subgroup of a word hyperbolic group is a finite index subgroup of only finitely many other subgroups.

1. Introduction.

Introductory remarks. There has been much interest recently in reintroducing geometric ideas into group theory. One of the most interesting examples is Gromov's work on the class of word hyperbolic groups [Gr]. The notion of a word hyperbolic group arose in the work of J. Cannon [C] and M. Gromov [Gr] as a generalization of certain group theoretic properties of discrete groups of isometries of classical hyperbolic spaces $\mathbf{H}^n$, the origins of which are to be found in Dehn's solution of the word and conjugacy problems for surface groups (see [De]). Word hyperbolic groups are finitely presented, and examples are provided by finite groups, finitely generated free groups, fundamental groups of compact surfaces (except for the torus and the Klein Bottle), and groups which act properly discontinuously and cocompactly on hyperbolic space of any dimension. Finite extensions of word hyperbolic groups, and free products of finitely many word hyperbolic groups are also word hyperbolic, though a direct product of two infinite word hyperbolic groups is not word hyperbolic. We will give some background information about word hyperbolic groups in Sections 2 and 3 and refer the reader to Gromov's original article [Gr], and the several commentaries that now exist (for instance [ABC], [Bo1], [CDP], [GH]) for further information, and proofs of the basic results.

The object of this note is to prove for word hyperbolic groups analogues of a theorem of Greenberg for finitely generated subgroups of Fuchsian groups (see Theorem 1, Part (2) below). In our case these results concern *quasiconvex subgroups* (for the full definition see Section 2), which are a special kind of finitely generated subgroup. Indeed, in a free group a subgroup is finitely generated if and only if it is quasiconvex. In a word hyperbolic group, all finite subgroups, all cyclic subgroups, and all subgroups of finite index are quasiconvex. Also, a quasiconvex subgroup of a word hyperbolic group

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is word hyperbolic. It is not true in general that a finitely generated subgroup of a word hyperbolic group is word hyperbolic as there exist finitely generated subgroups of word hyperbolic groups which are not finitely related (see for instance [R], [BMS]). Moreover, recently N. Brady [Br] constructed a remarkable example of a torsion-free hyperbolic group which contains a finitely presented subgroup which is not itself word hyperbolic.

An important part of Gromov's work involves the introduction of a "boundary" which compactifies the Cayley graph of a word hyperbolic group (in fact it compactifies a δ-hyperbolic space). This construction generalizes the usual compactification of n-dimensional hyperbolic space by a $n-1$ dimensional sphere. A similar construction for groups was also investigated by Floyd in his thesis (see [F]).

In Section 2 of this paper we define a very rudimentary form of boundary for the group, which is not in fact a compactification. We show that several results about word hyperbolic groups and their subgroups can be proved using exactly the same techniques which are used in classical hyperbolic geometry, provided certain preparatory work is done. We introduce the notions of a boundary of a subgroup of a group, and of the convex hull of a subset of the boundary. We then show in 2.4(i) that for a quasiconvex subgroup A, the quotient of the convex hull of the boundary of A by the action of A is finite. Using this, we obtain:

THEOREM 1. *Let G be a word hyperbolic group, and let A be an infinite quasiconvex subgroup.*

(1) *If B is an infinite quasiconvex subgroup of G, and $A \cap B$ has finite index in A and in B, then $A \cap B$ has finite index in $A \vee B$, the subgroup generated by A and B in G.*

(2) *The subgroup A has finite index in only finitely many distinct subgroups of G.*

(3) *The subgroup A has finite index in its virtual normalizer*

$$VN_G(A) = \{g \in G \mid [A : A \cap gAg^{-1}] < \infty \text{ and } [gAg^{-1} : A \cap gAg^{-1}] < \infty\}.$$

In Section 3 we study the boundary of a δ-hyperbolic space (for instance the Cayley graph of a word hyperbolic group) as defined by Gromov, as a compactification of the space. The subgroup structure of word hyperbolic groups is now discussed from the point of view of understanding their limit sets in the boundary.

We improve on the results of Section 2, and show (Proposition 3.4) that an infinite subgroup is quasiconvex if and only if the quotient of a convex hull (appropriately defined) of its limit set by the action of the subgroup has finite diameter.

It is not possible to use the standard notion of convexity and convex hulls when working with word hyperbolic groups. Indeed, as it was demonstrated in a remarkable example of M. Bridson and G. Swarup [BS], it can happen that a (classical) convex hull of a finite subset of a Cayley graph of a group coincides with the whole group. Thus certain adjustments have to be made and an appropriate substitution for the classical notion of convex hull should be used.

Our new characterization of quasiconvex subgroups of word hyperbolic groups stresses the similarity between quasiconvexity and geometrical finiteness for classical hyperbolic groups and immediately yields the following result of G.Swarup [Swa].

THEOREM 2. *Let G be a torsion-free geometrically finite group of isometries of $\mathbb{H}^n$ without parabolics. Then G is word hyperbolic and a subgroup A of G is quasiconvex in G if and only if it is geometrically finite.*

We also obtain a criterion of commensurability for quasiconvex subgroups of word hyperbolic groups in terms of limit sets (Lemma 3.8) and obtain another proof of Theorem 1 as well as

THEOREM 3. *Let A be an infinite quasiconvex subgroup of a word hyperbolic group G and let B be a subgroup of G containing A. Then A has finite index in B if and only if A contains an infinite subgroup C which is normal in B.*

In Section 4 we remark that it is possible to give a criterion of quasiconvexity similar to Proposition 3.4 in terms of the elementary boundary we use in Section 1.

Theorem 3 has also been obtained by Mihalik and Towle [MT] and results similar to those of Section 3 were independently obtained by E. Swenson [Swe] and R. Gitik, M. Mitra, E. Rips and M. Sageev [GMRS]. Parts (1) and (2) were originally proved for finitely generated subgroups of free groups by L. Greenberg [G], using the usual compactification of the hyperbolic plane by a circle. He showed that part of the boundary can be naturally associated to a subgroup A of a discrete subgroup G of $PSL(2,\mathbb{R})$. He then showed that two subgroups have the same associated boundary if and only if they are finite extensions of a common subgroup.

We would also like to bring to the reader's attention Stallings' beautiful paper "The topology of finite graphs" [St] where most of Theorem 1 is proved for finitely generated subgroups of free groups.

Some comments on Theorem 1. Before proceeding to the definitions and proofs of our main theorem, we give some examples to illustrate the necessity of some conditions concerning infiniteness and quasiconvexity. Notice that a finite subgroup is always quasiconvex.

Statement (3), may fail if H is finite or if H is infinite but not quasiconvex. Consider the group $Q = \langle a, b \mid [a,b]^2 = 1 \rangle$ and $H = \{1\}$. Then Q is word hyperbolic (being a small cancellation group) and there are infinitely many distinct conjugates of the subgroup $M = gp([a,b]) \cong \mathbb{Z}_2$ of Q (by the usual small cancellation arguments). Thus H is a subgroup of finite index in infinitely many subgroups of G although H is quasiconvex in Q.

When G contains an infinite cyclic subgroup, the trivial subgroup $H = \{1\}$ is contained in infinitely many cyclic subgroups (each of which is quasiconvex when G is word hyperbolic). Concerning Theorem 3, there is a word hyperbolic group G which has a normal subgroup H isomorphic to the fundamental group of a closed hyperbolic surface with the quotient group Greenberg's theorem for quasiconvex subgroups of word hyperbolic groups G/H isomorphic to $\mathbb{Z}$. Again, in this case, the normalizer of H is all of G, and $|G : H| = \infty$.

One can also construct an example of an infinite, finitely generated (but not quasiconvex) subgroup H of a word hyperbolic group G which has finite index in infinitely many subgroups of G, as follows. Let Q be the one-relator group with torsion defined above. Now apply the construction of E. Rips [R] to obtain a short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where G is a torsion-free word hyperbolic group (as it satisfies a small cancellation condition), and K is an infinite, normal two-generator subgroup of G. Let q be a conjugate of $[a,b]$ in Q and g be an element in the preimage of q in G. Put $K_q = gp(K,g)$. Then $g^2 \in K, g \notin K$ and K is a subgroup of index 2 in K_q. Moreover, the image of K_q in Q is the cyclic group of order two $gp(q)$. Thus for different conjugates q of g we get different subgroups K_q of G containing K as a subgroup of finite index.

Hyperbolic groups with all finitely generated subgroups quasiconvex. We say that a word hyperbolic group G belongs to class (Q) when a subgroup A of G is quasiconvex in G if and only if A is finitely generated.

Among the groups known to belong to the class (Q) are finitely generated free groups and fundamental groups of closed hyperbolic surfaces (see [Sho] and [Pi]). It is also clear that finite extensions and subgroups of finite index (in fact all quasiconvex subgroups) of groups in (Q) belong to (Q). G. Swarup has shown [Swa] that if G is a geometrically finite torsion-free Kleinian group without parabolics with a nonempty discontinuity domain (*i.e.* the limit set of G is not the whole sphere S^2) then G is in (Q). The last observation is based on a result of W. Thurston [Mo].

Notice that any group G in the class (Q) has the Howson property, that is the intersection of any two finitely generated subgroups is finitely generated (indeed, the intersection of two quasiconvex subgroups is quasiconvex and so is finitely generated, finitely presentable and even word hyperbolic). This approach to Howson property is discussed in more detail in [Sho].

In the class (Q), Theorem 1 can be stated in terms of finitely generated subgroups. In [Ba] H. Bass proved Theorem 1(2) in these terms for the case when G is the fundamental group of a finite graph of groups $\mathbb{Y}$ with finite vertex groups. Such a group G lies in (Q) since it is virtually free. There are two natural proper hyperbolic spaces on which G admits a properly discontinuous cocompact isometric action. One is the Cayley graph of G with respect to some finite generating set. Another is the Bass-Serre universal covering tree for the graph of groups $\mathbb{Y}$. The result can be established for G by studying either of these actions. The original proof of H. Bass employed the second action but the ideas involved in it are quite similar to ours.

In the case when G is a free group of finite rank, the Bass-Serre tree coincides with the Cayley graph of G with respect to a free basis. This action was studied by J. Stallings in [Sta] to obtain the same statement for free groups. The original proof of this result by L. Greenberg (which also applies to hyperbolic surface groups) used the fact that a free group G admits a discrete properly discontinuous geometrically finite action in $\mathbb{H}^2$ and therefore the action of G on the (classical) convex hull of its limit set is properly discontinuous and cocompact.

From our point of view the proof given in the present note captures the similarity of all these approaches.

2. **Definitions and Proof of Theorem 1.**

Some definitions. If (X, d) is a metric space and I is a connected subset of a real line, a map $f: I \longrightarrow X$ is called *geodesic* if $d\big(f(t), f(s)\big) = |t-s|$ for any $s, t \in I$. Sometimes the image of this map is also termed a geodesic. A metric space is termed *geodesic* if there is at least one geodesic joining each pair of points.

If Δ is a geodesic triangle (that is a triangle with geodesic sides) in a metric space (X, d) with vertices x, y, z and geodesic sides $[x,y]$, $[x,y]$, $[y,z]$ then there are unique points s, q, r on the sides $[x,y]$, $[x,z]$ and $[y,z]$ such that $d(x,s) = d(x,q)$, $d(z,q) = d(z,r)$ and $d(y,s) = d(y,r)$. The points s, q, r are called the *vertices* of the inscribed triangle in the triangle Δ. The triangle Δ is said to be *δ-slim* if, for any points s' on $[x,y]$ and q' on $[x,z]$ chosen such that $d(x,s') = d(x,q') \leq d(x,s) = d(x,q)$, we have $d(s',q') \leq \delta$ (and the analogous condition holds interchanging x, y and z).

A geodesic metric space (X, d) is called *δ-hyperbolic* if all geodesic triangles are δ-slim.

Let G be a group generated by a finite set S. We define the *S-length* of an element $g \in G$, denoted by $l_S(g)$, as the minimal number t such that g can be expressed in the form $g = s_1 \cdots s_t$ where $s_i \in S^{\pm 1}$. A minimizing expression is termed an *S-geodesic* representative of g. We will often confuse formal words in the alphabet S with the elements they represent in G.

The empty word e by convention represents the identity element of G. Define $d_S(g_1, g_2) = l_S(g_1^{-1} g_2), g_1, g_2 \in G$. This defines a metric on G called the *word metric* associated to S. The word metric d_S can be extended to a metric (also denoted d_S) on the Cayley graph $\Gamma(G, S)$, by declaring each edge to be isometric to the unit interval. This metric is clearly invariant under the natural left action of G.

A finitely generated group G is called *word hyperbolic* if for some finite generating set S of G there is $\delta \geq 0$ such that all geodesic triangles in the Cayley graph $\big(\Gamma(G, S), d_S\big)$ of G are δ-slim. (It turns out that this property is independent of the finite generating set chosen—see [Gr], [ABC], etc.).

In general, a subset A of a geodesic metric space (X, d) is termed ϵ-quasiconvex in X if any geodesic $[a_1, a_2]$ in X joining points $a_1, a_2 \in A$ is contained in an ϵ-neighbourhood of A. If G is a finitely generated group and A is a subgroup of G, we say that A is *quasiconvex* in G if for some finite generating set S of G A is a quasiconvex subset of the Cayley graph $\Gamma(G, S)$.

The following properties of quasiconvex subgroups and word hyperbolic groups and their quasiconvex subgroups are of importance to us here:

PROPOSITION 2.0. *Let G be a finitely generated group, and S a finite set of generators for G.*

(1) *[GS] If A is a quasiconvex subgroup of G then A is finitely generated.*
(2) *[GS] If A is a quasiconvex subgroup of G then for any finite generating set T of A and for any finite generating set S of G there is a constant $B \geq 0$ such that $d_T(a_1, a_2) \leq \mathrm{B}d_S(a_1, a_2), a_1, a_2 \in A$.*
(3) *[Sho] If $A_1, \dots, A_k$ are quasiconvex subgroups of G, then $A = A_1 \cap \cdots \cap A_k$ is quasiconvex in G.*

If in addition, G is a word hyperbolic group, then:

(4) *[BGSS, Theorem 3.1] If A is a finitely generated subgroup of G and for some finite generating set T of A and some finite generating set S of G there is a constant $B \geq 0$ such that $d_T(a_1, a_2) \leq \mathrm{B}d_S(a_1, a_2), a_1, a_2 \in A$, then A is a quasiconvex subset of $\Gamma(G, S)$. (Together with (2) this implies that the definition of quasiconvexity for a subgroup of a word hyperbolic group G does not depend on the choice of a finite generating set of G).*
(5) *[CDP, Chapter 3, Proposition 4.1] If A is a quasiconvex subgroup of G then for any finite generating set T of A and for any finite generating set S of G A is a quasiconvex subset of $\Gamma(G, S)$.*
(6) *[ABC, Lemma 3.8] If A is a quasiconvex subgroup of G then A is also word hyperbolic.*
(7) *If A, B are subgroups of G such that A has finite index in B, then A is quasiconvex in G if and only if B is quasiconvex in G.*
(8) *[GH, Chapter 4, Theorem 13] There are only finitely many conjugacy classes of elements of finite order in G.*
(9) *[GH, Chapter 8, Theorem 37] An infinite subgroup of G contains an element of infinite order.*
(10) *[ABC, Corollary 3.4] If C is a virtually cyclic subgroup of G then C is quasiconvex in G.*
(11) *[N] If $\phi \in \mathrm{Aut}(G)$ and A is a subgroup of G, then A is quasiconvex in G if and only if $\phi(A)$ is quasiconvex in G.*

Other interesting properties of word hyperbolic groups, for instance that G is finitely presented and has a linear isoperimetric function, will not be needed here. For proof of these results and further information about quasiconvexity and word hyperbolic groups, we refer the reader to [Sho], [Gr], [ABC], [GH] and the other hyperbolic references.

An elementary boundary for a finitely generated group. In this section G is a finitely generated group, and S is a finite set of generators. The group G acts by left multiplication on the Cayley graph $\Gamma_S(G)$, by isometries. We construct a 'boundary' for G, which is essentially the boundary defined by Gromov for word hyperbolic groups deprived of its topology.

Consider the set of subsemigroups $\mathfrak{S} = \left\{\{g^i \mid i > 0\}, g \in G\right\} =_{def} \left\{\{g^+\} \mid g \in G\right\}$.

We now extend to a larger set (where there is an obvious left action induced by left multiplication), and define $\Sigma = G \times \mathfrak{G} = \big\{\{hg^i\}, h, g \in G\big\} = \{hg^+, h, g \in G\}$.

We define an equivalence relation on Σ by: $hg^+ \sim h'g'^+$ if $\{hg^i; i > 0\}$ lies in a bounded neighbourhood of $\{h'g'^j; j > 0\}$ and $\{h'g'^j; j > 0\}$ lies in a bounded neighbourhood of $\{hg^i; i > 0\}$.

This is clearly an equivalence relation, and we denote the equivalence class of hg^+ by $[hg^+]$ (and $[hg^-] = [h(g^{-1})^+]$). Notice that $[g^+] = [g'^+]$ if and only if g and g' have a common (positive) power: there is some word d and infinitely many pairs (i,j) of positive numbers such that $g^i = g'^j d$.

The elementary boundary of G is defined to be $\tilde{\partial}G = (\Sigma/\sim) - \{[1]\}$.

We remove the equivalence class of the identity element due to its special nature—it clearly contains all elements of finite order. One consequence of this is that a torsion group will have empty boundary.

It is clear from the construction that G acts on $\tilde{\partial}G$ on the left by $h \cdot [h'g^+] = [hh'g^+]$.

LEMMA 2.1. *(i) If* $[g^+] = [(g^-]$ *then* g *has finite order in* G*, and* $[g^+] = [1]$*;*

(ii) The left action of G *on* $\tilde{\partial}G$ *by left multiplication is the same as the action induced by conjugation* $h \cdot [h'g^+] = [(hh'h^{-1})(hgh^{-1})^+]$.

(iii) The definition of $\tilde{\partial}G$ *is independent of the choice of finite generating set.*

PROOF. (i) If the positive powers of G lie in a bounded neighbourhood of the negative powers of g, then there is some element d in G such that $g^i = g^{-j}d$, and $g^k = g^{-\ell}d$ for some $i, j, k, \ell > 0$. It follows that $d = g^{i+j} = g^{k+\ell}$ in the group G. As this happens for infinitely many such quadruples, g has finite order.

(ii) Notice that $d(hg^ih^{-1}, hg^i) = \ell_S(h)$, so that $\{(hh'h^{-1}) \cdot hg^i = hh'g^i \mid i > 0\}$ lies in a $\ell_S(h)$-neighbourhood of $\{(hh'h^{-1}) \cdot hg^ih^{-1} = hh'g^ih^{-1} \mid i > 0\}$ and vice versa.

(iii) Choosing a finite set of generators Y for G changes length in the group by at most a constant factor, so that equivalence is unchanged.

Notice that as a result of part (ii) of this lemma, in each equivalence class there is a representative of the form $\{g^+\}$, *i.e.* $\Sigma/\sim = \mathfrak{G}/\sim$. We now study subsets of this boundary which correspond to subgroups.

DEFINITION. Let A be a subset of G. We define the boundary of A in G, to be: $\tilde{\partial}_G A = \{[a^i] \mid a \in A\}$. Notice that, by (ii) above, $\tilde{\partial}_G G = \tilde{\partial}G$.

For a subset Y of the boundary $\tilde{\partial}G$, we define the *stabilizer* of Y to be:

$$\mathrm{Stab}_G(Y) = \{g \in G \mid gY = Y\}.$$

For a fixed finite set S of generators for G, define the *convex hull* of Y to be: $\mathcal{H}_S(Y) = \{v \in G$ such that there are $g, g', h, h' \in G$ such that $[hg^+], [h'g'^+] \in Y$, and there are two infinite, strictly increasing sequences n_i, m_i, such that v lies on a S-geodesic from hg^{n_i} to $h'g'^{m_i}\}$.

Notice that, while the definition of the stabilizer is independent of the choice of generating set, the convex hull *does* depend on the choice. We will however refer to Stab(H) and $\mathcal{H}(Y)$ when the ambient group and generating set are clear from the context.

EXAMPLES. 0. By Lemma 2.1(i) above, the boundary of a finite group, (or indeed of an infinite torsion group) is empty.

1. The boundary of the infinite cyclic group has two elements, (which we can think of as $\pm\infty$), and the stabilizer of each of these elements is the entire group. With the single generator 1, the convex hull of the boundary is clearly the whole group, as is the case for any finite set of generators.

2. The free abelian group of rank two: each maximal cyclic subgroup has a naturally associated "slope", and each rational number $\frac{m}{n}, n > 0$ (and $0, \infty$) defines a cyclic subgroup $\langle x^m y^n \rangle$. The boundary of this cyclic subgroup is two points, and the boundary of the group is a countable set (of rational slopes). All rank two subgroups have the same boundary, that of the whole group. Notice that the convex hull of the boundary of any nontrivial subgroup is the whole group.

3. The free group of rank two: each cyclically reduced element g which is not a proper power has its own equivalence class $[g^+]$. The Cayley graph with respect to a set of free generators is a tree, and the convex hull of the boundary of a cyclic subgroup generated by a cyclically reduced word g is the 'axis'—the line containing the elements of this subgroup (*i.e.* vertices corresponding to initial segments of powers of g).

For a noncyclically reduced word, the convex hull is the translated axis of the cyclically reduced part; if $w = vuv^{-1}$ with u cyclically reduced, then the elements of $\mathcal{H}(\tilde{\partial}\langle w \rangle)$ are the vertices of the Cayley graph corresponding to vu', where u' is an initial segment of the word u^m, for $m \in \mathbb{Z}$.

LEMMA 2.2. *Let G be a finitely generated group, S a finite set of generators and $Y \subset \tilde{\partial}G$.*

(i) $\mathrm{Stab}(Y)$ acts freely on the convex hull $\mathcal{H}_S(Y)$.

(ii) If A is a subgroup of $G, A \subset H_G(A) \subset \mathrm{Stab}(\tilde{\partial}_G A)$.

(iii) If A and B are subgroups of G, and A has finite index in B, then $\tilde{\partial}_G A = \tilde{\partial}_G B$.

PROOF. (i) G acts freely on itself on the left. It suffices to show that if $v \in \mathcal{H}_S(Y)$, and $b \in \mathrm{Stab}(Y)$, then $bv \in \mathcal{H}_S(Y)$.

Suppose that v lies on geodesics from hg^i to $h'g'^j$, $[hg^+], [h'g'^+] \in Y$. For $b \in G$, bv lies on a geodesic from bhg^i to $bh'g'^j$, and for $b \in \mathrm{Stab}(Y)$, it follows that $bv \in \mathcal{H}_S(Y)$.

(ii) As in Lemma 2.1 (ii) above, for $a \in A, d(ag^i, ag^ia^{-1}) \leq |a|$, so $\{ag^i\} \sim \{ag^ia^{-1}\}$. In fact, as long as a normalizes A, this is an element of $\tilde{\partial}_G A$.

(iii) If $b \in B$, then $b^k = a$ for some $a \in A, k > 0$, so $[b^+] = [a^+]$. ∎

(Notice that in the proof of (iii) we did not use the fact that k can be chosen to be bounded. Thus it was actually proved that if A is a subgroup of B and for each element of B some power of it belongs to A, then $\tilde{\partial}_G A = \tilde{\partial}_G B$).

We now concentrate on word hyperbolic groups and their quasiconvex subgroups, which have particularly nice properties. Recall that all cyclic subgroups in a word hyperbolic group, are quasiconvex.

LEMMA 2.3. *Let A be a quasiconvex infinite cyclic subgroup of the finitely generated group G. Then:*

(i) $\mathcal{H}(\tilde{\partial}_G A)$ is infinite;

(ii) if, in addition, G is a word hyperbolic group, then $\mathcal{H}(\tilde{\partial}_G A)$ is contained in a bounded neighbourhood of A, and is thus quasigeodesic too.

PROOF. Fix a finite set of generators S for G.

(i) Let $\langle c \rangle$ be a K-quasiconvex infinite cyclic subgroup. We show that *all* geodesics from c^m to c^{-m} pass through a ball of bounded radius about the identity vertex, for all m sufficiently large. Notice that these geodesics all lie in a K neighbourhood of the quasiconvex subgroup $\langle c \rangle$.

Consider a geodesic triangle with vertices $1, c^m, c^{-m}$. Let v_1, v_2 be adjacent vertices on the geodesic α joining c^m and c^{-m} such that there are positive numbers $p, q > 0$ with $d(v_1, c^{-p}) \leq K$ and $d(v_2, c^q) \leq K$. Then $d(c^{-p}, c^q) < 2K + 1$; this bounds $p + q$, and in particular bounds p and q. Hence the geodesic α passes through a bounded neighbourhood B of the identity element. It follows that there is an infinite sequence $\{m_i\}$ and geodesics γ_i joining c^{m_i} and c^{-m_i}, and a vertex $v \in B$, such that every γ_i passes through the vertex v. Thus $H_S(\tilde{\partial}_G A)$ is nonempty, and by 2.2(i) it is therefore infinite.

(ii) Suppose that $v \in \mathcal{H}_S(\tilde{\partial}_G A)$. Let $\{hg^i\}$ and $\{h'g'^j\}$ be sequences representing elements of $\tilde{\partial}_G A$ such that v lies on infinitely many geodesics joining the points hg^i and $h'g'^j$. There is a constant $C > 0$, and $a, b \in A$ such that $\{hg^i \mid i > 0\}$ and $\{h'g'^j \mid j > 0\}$ are contained in C neighbourhoods of the cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$.

Choose i, j sufficiently large so that $\min\{d(v, g^i), d(v, g'^j)\} > 2\delta + C$. Let $a', b' \in A$ be nearest elements of A to hg^i and $h'g'^j$.

By studying the quadrilateral with geodesic sides and vertices $a', b', hg^i, h'g'^j$, we see that the vertex v is at distance at most 2δ from some point on a geodesic joining a' and b'. As A is quasigeodesic with constant K, this means that v is at distance at most $K + 2\delta$ from A. ∎

COROLLARY 2.4. *Let A be a quasiconvex subgroup of the word hyperbolic group G.*

(i) The quotient space $\mathcal{H}_X(A)/A$ is finite.

(ii) A has finite index in $\operatorname{Stab}(\tilde{\partial}_G A)$.

PROOF. As $\mathcal{H}(A)$ lies in a bounded K-neighbourhood of A, and A acts freely on $\mathcal{H}(A)$, each element of the quotient space has a representative of length at most K. These represent the cosets of A in $\operatorname{Stab}(\tilde{\partial} A)$. ∎

It is now easy to deduce Theorem 1:

PROOF OF THEOREM 1. (1) If $A \cap B$ has finite index in A, and A is quasiconvex in G, then $A \cap B$ is also quasiconvex in G. By Lemma 2.2 it follows that $\tilde{\partial}_G A = \tilde{\partial}_G(A \cap B) = \tilde{\partial}_G B$. As A and B both stabilize $\tilde{\partial}_G(A \cap B)$, so does $A \vee B$. Hence $A \cap B < A \vee B < \operatorname{Stab}\big(\tilde{\partial}_G(A \cap B)\big)$, and applying 2.4(ii) and the quasiconvexity of $A \cap B$ gives the result.

(2) If A has finite index in B, then $\tilde{\partial}_G A = \tilde{\partial}_G B$, and $A < B < \operatorname{Stab}(\tilde{\partial}_G A) = C$. If A is quasiconvex then B is a finite index subgroup of C, of index at most $[C : A]$.

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There are only finitely many such subgroups. In fact, if $\{1, g_1, \ldots, g_k\}$ is a set of coset representatives for A in C, then B is obtained by adjoining some subset of the $\{g_i\}$ to A.

(3) Suppose that $g \in VN_G(A)$. Put $U = gAg^{-1} \cap A$. Since U is of finite index in A and in gAg^{-1}, $K = \tilde{\partial}_G(U) = \tilde{\partial}_G(A) = \tilde{\partial}_X(gAg^{-1})$. However $\tilde{\partial}_X(gAg^{-1})$ is equal to the set $\{g[a^+] \mid a \in A\}$ that is to gK. Thus $gK = K$ and $g \in \mathrm{Stab}_G(K)$. Since $g \in VN_G(A)$ was chosen arbitrarily, we conclude that $A \leq VN_G(A) \leq \mathrm{Stab}_G(K)$. By Corollary 2.4(ii) this implies that A has finite index in $VN_G(A)$.

Notice that part (2) of the theorem gives another way to see that $\langle a, b \mid b^{-1}ab = a^2\rangle$ is not word hyperbolic. The subgroup $\langle a\rangle$ has index 2 in $\langle bab^{-1}\rangle$, and index 2^n in $\langle b^n ab^{-n}\rangle$.

3. **Gromov boundary and quasiconvexity in hyperbolic groups.**

Definitions and notations. In order to obtain stronger results than those obtained in the Section 2, we require a boundary with a topological structure. Such a boundary has been defined by Gromov for δ-hyperbolic spaces (see also Floyd's work [F]). For the detailed discussion and proofs of the basic properties of this compactification (for instance that the elementary boundary of Section 2 can be viewed as a dense subset), the reader is referred to [Gr, Chapter 1,3,7], [CDP, Chapter 2], [GH, Chapter 5-7] and [ABC, Chapter 4]. We here give some of the basic definitions that we shall need.

A metric space (X, d) is said to be *proper* if all closed metric balls in X are compact. Any proper δ-hyperbolic metric space (X, d) can be compactified by adding points of the *ideal boundary* (or visual boundary) ∂X to X. Each point of the boundary ∂X is an equivalence class of geodesic rays $r\colon [0, \infty) \longrightarrow X$ where rays r_1 and r_2 are equivalent if $\sup\{d(r_1(t), r_2(t))\} < \infty$. It is not hard to see that if r_1 and r_2 are equivalent then $\sup\{d(r_1(t), r_2([0,\infty)) \mid t \in [0,\infty)\} \leq \max\{d(r_1(0), r_2(0)), 2\delta\}$.

There is another way to think about points of the boundary. Fix a point $p \in X$ and put

$$(x,y)_p = (1/2)\big(d(x,p) + d(y,p) - d(x,y)\big)$$

for any $x, y \in X$. Notice that, considering a geodesic triangle with vertices p, x, y, we see that $(x,y)_p$ is equal to the distance from p to the two vertices of the inscribed triangle which lie on the sides $[p,x]$ and $[p,y]$ (see the introduction to Section 2).

We say that a sequence $(x_i)_{i\in\mathbb{N}}$ of points in X *converges to infinity* if

$$\lim_{n\to\infty}\{\inf_{i,j\geq n}(x_i, x_j)_p\} = \infty.$$

Two sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ converging to infinity are said to be equivalent if

$$\lim_{n\to\infty}\{\inf_{i,j\geq n}(x_i, y_j)_p\} = \infty.$$

One can check that the last two definitions do not depend on the choice of a basepoint $p \in X$. It follows from the definition that if $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are sequences of points in X such that for some finite K, $d(x_i, y_i) \leq K$ for each $i \in \mathbb{N}$ then $(x_i)_{i\in\mathbb{N}}$ converges to infinity if and only if $(y_i)_{i\in\mathbb{N}}$ converges to infinity, and the sequences are equivalent in this case.

The points of the ideal boundary ∂X can be identified with equivalence classes of sequences converging to infinity. If a is the equivalence class of $(x_i)_{i\in\mathbf{N}}$ we write $\lim_{i\to\infty} x_i = a$. It is obvious that if $r\colon [0,\infty) \to X$ is a geodesic ray then the sequence $x_i = r(i)$ converges to infinity. It is also clear that if r_1, r_2 are equivalent geodesic rays then the sequences $\big(r_1(i)\big)_{i\in\mathbf{N}}$ and $\big(r_2(i)\big)_{i\in\mathbf{N}}$ are also equivalent.

Moreover, since (X,d) is proper, any sequence converging to infinity is equivalent to the sequence $\big(r(i)\big)_{i\in\mathbf{N}}$ for some geodesic ray r. This gives an identification between the points of ideal boundary and equivalence classes of sequences converging to infinity [Gr, 7.5].

It can be shown (see [CDP, Chapter 2, Theorem 2.1]) that for any two distinct points $a, b \in \partial X$ there exists a (not necessarily unique) bi-infinite geodesic $r\colon (-\infty,\infty) \to X$ such that $\lim_{i\to\infty} r(-i) = a$ and $\lim_{i\to\infty} r(i) = b$. This geodesic r is sometimes denoted by (a,b) and referred to as a geodesic joining a and b.

Analogously, for any point $a \in \partial X$ and for any point $x \in X$ there is a geodesic ray $r\colon [0,\infty) \to X$ such that $r(0) = x$ and $\lim_{i\to\infty} r(i) = a$. Such a ray r is referred to as a geodesic joining x and a and is sometimes denoted $[x, a)$.

Fix a basepoint p of X and let $r_1, r_2\colon [0,\infty) \to X$ be geodesic rays joining p to points a and b in the boundary. We put

$$(r_1, r_2)_p = \sup_{t\in[0,\infty)} \big(r_1(t), r_2(t)\big)_p = \lim_{t\to\infty} \big(r_1(t), r_2(t)\big)_p.$$

We define $(a,b)_p$ to be the infimum of the numbers $(r_1,r_2)_p$ where r_1, r_2 vary over all geodesic rays joining p to a and b. In fact it can be shown that for any such r_1 and r_2, we have $|(r_1,r_2)_p - (a,b)_p| \leq 10\delta$.

The boundary ∂X of X is topologized by putting $\lim_{i\to\infty} a_i = a$ if $\lim_{i\to\infty}(a_i, a)_p = \infty$, where $a, a_i \in \partial X$.

It is easy to describe this topology in the language of sequences converging to infinity. Namely, let a_n, $n = 1, 2, \ldots$, be the equivalence class of a sequence $(x_{n,i})_{i\in\mathbf{N}}$ and let a be the equivalence class of a sequence $(x_i)_{i\in\mathbf{N}}$. Then $\lim_{i\to\infty} a_n = a$ if

$$\lim_{m\to\infty} \big\{ \inf_{i,j\geq n} (x_i, x_{m,j})_p \big\} \to \infty.$$

All the definitions given above do not depend on the choice of a basepoint. Moreover ∂X and $X \cup \partial X$ are compact with the topologies described above [CDP, Chapter 2, Proposition 3.2].

We need one more piece of notation. If $x \in X$ and r is a geodesic ray joining p with a point $a \in \partial X$, we put $(x, r)_p = \lim\big(x, r(t)\big)_p = \sup\{\big(x, r(t)\big)_p \mid t \in [0,\infty)\}$. One observes that if a, b are distinct points in ∂X, and points $x \in r_1 = [p, a), y \in r_2 = [p, b)$ are chosen such that $M = (x,b)_p = (y,a)_p$, and $d(x,p) > 50(M+1)\delta$, $d(y,p) > 100(M+1)\delta$, then $(x,y)_p$, $(x, r_2)_p$, and $\big(r_1, y)_p\big)$ are within 10δ of $(r_1, r_2)_p$.

Convex hulls and limit sets in hyperbolic spaces. Let X be a proper δ-hyperbolic space for some $\delta > 1$.

We need the following obvious lemma.

LEMMA 3.1. *Any ideal geodesic quadrilateral, that is a quadrilateral with vertices on the boundary of X, is a 100δ-quasiconvex subset of X.*

PROOF. This follows immediately from the fact that ideal quadrilaterals in X are 10δ-thin, that is any side is contained in a 10δ-neighborhood of the union of other sides.

We can now define the convex hull of a subset of the boundary. We can now use infinite rays where in Section 2 we had to make explicit reference to infinite families of geodesics.

DEFINITION. (see [Gr, 7.5.A]) Let X be a proper δ-hyperbolic space with $\delta > 1$, and let K be a subset of ∂X containing at least two points. Then the set $CK = \bigcup_{a,b \in K}(a,b)$ is called the *convex hull* of K in X.

LEMMA 3.2. *Let X and K be as above. Then CK is 100δ-quasiconvex in X.*

PROOF. Let $x, y \in CK$. Then $x \in (a,b)$ and $y \in (c,d)$ for some $a,b,c,d \in K$. Assume that the points a,b,c,d are distinct (it will be clear from the argument that the case when some of them coincide is completely analogous).

Join points b and c by a geodesic (b,c) and points d and a by a geodesic (d,a) in X. By definition of CK the ideal quadrilateral

$$D = (a,b) \cup (b,c) \cup (c,d) \cup (d,a)$$

is contained in CK. By Lemma 3.1 a geodesic (x,y) is contained in 100δ-neighborhood of D and so in 100δ-neighborhood of CK. Thus CK is 100δ-quasiconvex.

Let (X,d) be as above and suppose a group G acts on X by isometries so that the action is *properly discontinuous* (*i.e.* for any compact Q in X the set $\{g \in G \mid Q \cap gQ \neq \emptyset\}$ is finite) and *cocompact* (that is X/G with the quotient topology is compact). The action of G on X induces an action of G on ∂X by homeomorphisms: namely an element g of G takes the equivalence class of a geodesic ray r to the equivalence class of the ray gr. Equivalently, if $(a_i)_{i \in \mathbb{N}}$ is a sequence converging to infinity, then an element g of G takes its equivalence class to the equivalence class of the sequence $(ga_i)_{i \in \mathbb{N}}$ (which obviously also converges to infinity).

Pick a basepoint $p \in X$. For any subgroup $A \leq G$ define the *X-limit set* of A, denoted by $\partial_X(A)$, as the collection of all those $b \in \partial X$ which are the limits of some sequences of points from Ap (in the topology of $X \cup \partial X$). It is easy to see that this definition does not depend on the choice of the basepoint $p \in X$ and that $\partial_X(A)$ is always closed.

Analogous to the results of Section 2, the limit sets defined above have the following properties:

LEMMA 3.3. *Let (X,d) be a proper, δ-hyperbolic space, and G a group acting properly discontinuously and cocompactly by isometries on X. Let A and B be subgroups of G. Then:*

(1) The group G is word hyperbolic.

(2) $\partial_X(A) = \emptyset$ if and only if A is finite.

(3) *If $\langle c\rangle$ is an infinite cyclic subgroup of G then $\partial_X(\langle c\rangle)$ consists of two distinct points $c^+ = \lim_{i\to\infty} c^i$ and $c^- = \lim_{i\to\infty} c^{-i}$.*

(4) *If A is contained in B then $\partial_X(A) \subset \partial_X(B)$.*

(5) *If A has finite index in B then $\partial_X(A) = \partial_X(B)$.*

(6) *If A is infinite then $\partial_X(A)$ contains at least two distinct points.*

(7) *If A is infinite then the sets $\partial_X(A)$ and $C\partial_X(A)$ are A-invariant.*

PROOF. Statement (1) follows from the fact that in these circumstances, X and G are quasi-isometric (see for instance [ECHLPT, Theorem 3.3.6]) and so G is word hyperbolic as a group quasi-isometric to a proper hyperbolic metric space [GH, Chapter 5, Theorem 12]. Statements (4), (5) and (7) are immediate from the definitions, while (2) follows immediately from (6). Statement (3) follows from the fact that an infinite cyclic subgroup is quasiconvex in G (though this is not used in 2.1) and from [CDP, Chapter 3, Theorem 2.2]. Statement (6) is a consequence of Proposition 2.0(9) together with (3) and (4).

We give a sketch of a proof of (6) independent of Proposition 2.0(9), using Proposition 2.0(8), which is easier to prove.

If A has an element of infinite order, the statement is trivial. Suppose now that every element in A has finite order. Since there are only finitely many conjugacy classes of elements of finite order in G, we can find an integer N such that the order of any element of A divides N.

Fix a basepoint $p \in X$. Since Ap is infinite and $X \cup \partial X$ is compact, the set $\partial_X(A)$ contains at least one point q. Assume $q = \partial_X(A)$. Then $Aq = q$. Fix a geodesic ray $r = [p, q)$ in X. Let $A = \{a_i | i = 1, 2, \dots\}$ be an enumeration of A such that $a_i \neq a_j$ when $i \neq j$. Since $a_i q = q$ the rays $a_i r$ and r are 2δ-close far away from p. Thus the action of a_i on r far away from p is 4δ-close to a translation by t_i. Since the action of G on the set of vertices in X (and so in r) is free and A is infinite, we conclude that $\lim_{i\to\infty} t_i = \infty$. Thus for big enough i we have $t_i > 1000N\delta$. But for such i the action of a_i^N on r far away from p is $16N\delta$-close to a translation by Nt_i. This contradicts the fact that $a_i^N = 1$ and (6) is proved. ■

As the set $C\partial_X(A)$ is A-equivariant, the quotient space $C\partial_X(A)/A$ inherits a metric from the restriction of d to $C\partial_X(A)$: put $\bar{d}(Ax, Ay) = \inf\{d(a_1x, a_2y) \mid a_1, a_2 \in A\}$ for $x, y \in X$.

Clearly if $x, y \in X$ then $\inf\{d(a_1x, a_2y) \mid a_1, a_2 \in A\} = \inf\{d(x, ay) \mid a \in A\}$. Moreover, for any $x, y \in X$ $\inf\{d(x, ay) \mid a \in A\} = \min\{d(x, ay) \mid a \in A\}$. Indeed, let $M = \inf\{d(x, ay) \mid a \in A\}$. Then there is $a_0 \in A$ such that $d(x, a_0y) \leq M + 1$. Since the closed metric ball $B = B(y, 2M + 2)$ is compact and the action of G on X is properly discontinuous, the set $\{a \in A \mid d(a_0y, ay) \leq 2M + 2\}$ is finite. Therefore the set $\{a \in A \mid d(x, ay) \leq M + 1\}$ is finite and so $\inf\{d(x, ay) \mid a \in A\} = \min\{d(x, ay) \mid a \in A\}$. Thus whenever $\bar{d}(Ax, Ay) = 0$ we have $Ax = Ay$. Besides for any $z \in X$, $a_1 \in A$ $\bar{d}(Ax, Ay) = \min\{d(x, ay) \mid a \in A\} \leq d(x, a_1z) + \min\{d(ay, a_1z) \mid a \in A\} = \bar{d}(x, a_1z) + \bar{d}(Ay, Az)$. Since $a_1 \in A$ was chosen arbitrarily, this implies that $\bar{d}(Ax, Ay) \leq \bar{d}(Ax, Az) + \bar{d}(Az, Ay)$. Thus $(C\partial_X(A)/A, \bar{d})$ is indeed a metric space.

We are now in a position to prove an improved, metric version of Proposition 2.4 (the necessary condition has become an equivalent condition).

PROPOSITION 3.4. *Let G be a word hyperbolic group and A be a subgroup of G. The following conditions are equivalent*

(i) A is quasiconvex in G;

(ii) for any proper δ-hyperbolic space X and for any properly discontinuous cocompact action of G on X by isometries the quotient space $C\partial_X(A)/A$ has finite diameter;

(iii) for some proper δ-hyperbolic space X and some cocompact properly discontinuous action of G on X by isometries the quotient space $C\partial_X(A)/A$ has finite diameter.

Proposition 3.4 provides an alternative definition of quasiconvexity which highlights the connection of the theory of word hyperbolic groups with the theory of classical hyperbolic groups (*i.e.* discrete groups of isometries of $\mathbb{H}^n$).

Before proving Proposition 3.4, we must do a little preliminary work.

Recall that if $(X, d), (Y, d_1)$ are metric spaces, a map $f: X \longrightarrow Y$ is called a *K-quasi-isometry* of X if there is $K > 0$ such that for each $x_1, x_2 \in X$ $(1/K)\, d(x_1, x_2) - K \leq d_1\big(f(x_1), f(x_2)\big) \leq K\, d(x_1, x_2) + K$.

The map f is called a *K-quasi-isometry* of X *onto* Y if in addition the image $f(X)$ is *K-cobounded* in Y, that is, for each $y \in Y$ there is $x \in X$ such that $d_1\big(y, f(x)\big) \leq K$. In this latter case we say that the two spaces are quasi-isometric.

A K-quasi-isometry of a connected subset of the real line to Y is called a K-quasigeodesic. In a δ-hyperbolic space X, there is a constant ϵ, depending on X and K, such that if γ is a K-quasigeodesic joining two points x, y, and α is a geodesic joining the same points, then α is contained in a ϵ-neighbourhood of γ, and γ is contained in a ϵ-neighbourhood of α. For a proof of this essential result ("quasigeodecis are close to geodesics"), and more about quasi-isometries, see [Gr, Chapter 7], [GH, Chapter 5], [CDP, Chapter 4], etc.

A proper metric space (X, d) is said to be λ-quasigeodesic ($\lambda > 1$), if for any pair of points $x, y \in X$ there is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $d(x_i, x_{i+1}) \leq \lambda$ for $i = 0, \ldots, n-1$ and $|i - j| \leq \lambda\, d(x_i, x_j) + \lambda$ for any i, j.

The following lemma will be needed in the proof of 3.4 (it is analogous to Theorem 3.3.6 of [ECHLPT]).

LEMMA 3.5. *Suppose a group G acts by isometries on the λ-quasigeodesic space X properly discontinuously and cocompactly. Choose a basepoint $p \in X$ Then the following hold:*

(1) The group G is finitely generated.

(2) If S is a finite generating set of G then the map $f: (G, d_S) \longrightarrow (X, d)$ defined by $f: g \longmapsto gp$ is a quasi-isometry of the Cayley graph $\big(\Gamma_S(G), d_S\big)$ onto (X, d).

PROOF. We use $B(x, r)$ to denote the closed metric ball of radius r centered at x.

To see that G is finitely generated we follow the chain argument as in [ECHLPT, Theorem 3.3.6]. Put $M = \operatorname{diam}(X/G)$. Put $S = \{g \in G \mid B(p, M+\lambda) \cap gB(p, M+\lambda) \neq \emptyset\}$.

The set S is finite since $B(p, M+\lambda)$ is compact and the action of G is properly discontinuous.

We claim that S generates G. Indeed, let $g \in G$. Consider a λ-quasigeodesic sequence $p = x_0, x_1, \dots, x_n = gp$ connecting p and gp. Put $B = B(p, M)$. Clearly $GB = X$. Therefore for any $i = 1, \dots, n-1$ there is $g_i \in G$ such that $x_i \in g_i B$. Put $g_0 = 1, g_n = g$. Then $(x_{i+1}, g_i p) \leq d(x_{i+1}, x_i) + d(x_i, g_i p) \leq M + \lambda$ and $d(x_{i+1}, g_{i+1} p) \leq M$ for $i = 0, 1, \dots, n-1$. Thus $g_i^{-1} x_{i+1} \in B(p, M+\lambda) \cap B(g_i^{-1} g_{i+1} p, M)$ and so $h_i = g_i^{-1} g_{i+1} \in S$. But clearly $g = h_0 h_1 \cdots h_{n-1}$ and hence G is generated by S.

We will prove statement (2) of Lemma 3.5 for this particular choice of a finite generating set S of G. The general statement will then follow immediately from the fact that changing a finite generating set of a group induces a quasi-isometry of the word metrics.

Put $N = \max\{d(p, sp) \mid s \in S\}$. By definition the set S is symmetric that is $S = S^{-1}$. Let $g_1, g_2 \in G$. We need to estimate $d(g_1 p, g_2 p)$ in terms of $d_S(g_1, g_2)$. If $s_1 \cdots s_k$ is a d_S-geodesic representative for $g_1^{-1} g_2$ then put $y_0 = g_1 p, y_i = g_1 s_1 \cdots s_i p$ for $i = 1, \dots, k$. Then $d(y_i, y_{i+1}) = d(p, s_{i+1} p)$ and so $d(g_1 p, g_2 p) = d(y_0, y_k) \leq Nk = N\, d_S(g_1, g_2)$.

Now we need to estimate $d_S(g_1, g_2)$ in terms of $d(g_1 p, g_2 p)$. Consider a λ-quasigeodesic sequence $z_0 = g_1 p, \dots, z_n = g_2 p$ connecting $g_1 p$ with $g_2 p$ (instead of a geodesic path $[g_1 p, g_2 p]$ as it is done in [ECHLPT, 3.3.6]). Now for any $i = 1, \dots, n-1$ there is $f_i \in G$ such that $z_i \in f_i B(p, M)$. Put $f_0 = g_1, f_n = g_2$ and $h_i = f_i^{-1} f_{i+1}$. As before, $d(f_i p, f_{i+1} p) \leq 2M + \lambda$ and hence $h_i \in S$ for each $i = 0, 1, \dots, n-1$. Again $g_1^{-1} g_2 = h_0 h_1 \cdots h_{n-1}$ and so $d_S(g_1, g_2) \leq n \leq \lambda\, d(g_1 p, g_2 p) + \lambda$ since the sequence $z_0, \dots, z_n$ is λ-quasigeodesic. Thus

$$(1/N)\, d(g_1 p, g_2 p) \leq d_S(g_1, g_2) \leq \lambda\, d(g_1 p, g_2 p) + \lambda.$$

It remains to note that the set $f(G) = Gp$ is cobounded in G. But we have already seen that $GB = GB(p, M) = X$ and so for any $x \in X$ there is $gp \in Gp$ such that $d(x, gp) \leq M$. This completes the proof of Lemma 3.5.

We need to introduce one extra piece of notation before proving Proposition 3.4. If Y is a set and d_1, d_2 are real-valued functions on $Y \times Y$ we term them *equivalent* and write $d_1 \sim d_2$ if for some $C > 0$ we have $(1/C)\, d_1 - C \leq d_2 \leq C d_1 + C$. Also if Z is a set, Y is a set equipped with a real-valued function d on $Y \times Y$ and $h\colon Z \longrightarrow Y$ is a map, we term the function d_1 on $Z \times Z$ given by $d_1(z_1, z_2) = d\big(h(z_1), h(z_2)\big)$ the *h-pullback* of d and denote it $h_*(d)$.

Thus, for instance, two finite sets of generators for a finitely generated group give rise to equivalent distance functions.

PROOF OF PROPOSITION 3.4. (i) implies (ii): Let A be a quasiconvex subgroup of G and suppose that G acts on X as in (ii). If A is finite the the statement is obvious since $\partial_X(A)$ is empty. Assume now that A is infinite so that it has at least two distinct limit points in ∂X. Pick a basepoint $p \in X$ so that $p \in C\partial_X(A)$ and a finite generating set S

for G. As in Lemma 3.5, the map $f\colon (G, d_S) \longrightarrow (X, d)$ given by $f(g) = gp$ is a K-quasi-isometry for some $K > 0$. Consider a point $x \in C\partial_X(A)$; by definition $x \in (t_1, q_1)$ for some $t_1, q_1 \in \partial_X(A)$. This implies that for some points $t \in [x, t_1)$ and $q \in [x, q_1)$ far from x, and for some $a, b \in A$, the values $(t, ap)_x, (q, bp)_x$ are big.

More precisely, for some $L > 1000\delta$, the points t, q and $a, b \in A$ are such that $(t, ap)_x \geq L, (q, bp)_x \geq L$. Consideration of the quadrilateral formed by ap, bp, t and q, shows that there is a point $y \in [ap, bp]$ with $d(x, y) \leq 6\delta$.

Consider now a d_S-geodesic sequence of points $g_0 = a, g_1, \dots, g_n = b$ in G, that is $d_S(g_i, g_j) = |i - j|$. Since f is a K-quasi-isometry, the broken line $\alpha = [ap, g_1p] \cup [g_1p, g_2p] \cup \dots \cup [g_{n-1}p, bp]$ is a $2KM$-quasigeodesic where $M = \max\{d(p, sp), s \in S\}$. As quasigeodesics are close to geodesics, there is a constant $\epsilon = \epsilon(X, K, M)$ such that α is ϵ-close to the geodesic $[ap, bp]$, and so there is $k, 0 \leq k \leq n$ such that $d(y, g_kp) \leq M + \epsilon$. Therefore $d(x, g_kp) \leq 6\delta + M + \epsilon$. Now A is e-quasiconvex in (G, d_S) and $g_0 = a, g_1, \dots, g_n = b$ is a d_S-geodesic sequence. Therefore there is $a_1 \in A$ such that $d_S(g_k, a_1) \leq e$. This implies $d(g_kp, a_1p) \leq Ke + K$ and so $d(x, a_1p) \leq 6\delta + M + \epsilon + Ke + K = K_1$. This implies $\operatorname{diam}(C\partial_X(A)/A) \leq K_1$ since $x \in \partial_X(A)$ was chosen arbitrarily.

(ii) implies (iii): Let (X, d) be the Cayley graph of G with respect to some finite generating set. Then X is δ-hyperbolic for some $\delta > 1$, geodesic, has compact closed metric balls and the action of G on X by left multiplication is properly discontinuous and cocompact.

(iii) implies (i): Assume (X, d) and an action of G on X are as in (iii). Fix a finite generating set S for G and word metric d_S on G. If A is finite then it is obviously quasiconvex in G. Assume now that A is infinite. By Lemma 3.3, the boundary $\partial_X(A)$ contains at least two points and so $C = C\partial_X(A)$ is nonempty. Moreover C is 100δ-quasiconvex in X by Lemma 3.1. Therefore the closure (in X) $\bar{C}$ is 102δ-quasiconvex and A-invariant. It is easy to see that $(C, d|_C)$ is λ-quasigeodesic for $\lambda = 10000\delta$, as follows. Consider any two points $x, y \in \bar{C}$ and a geodesic segment $[x, y]$ in X. Choose a sequence $v_0 = x, v_1, \dots, v_n = y$ of points on $[x, y]$ so that $d(v_{n-1}, v_n \leq 1$ and $d(v_i, v_{i+1}) = 1$ for $i = 0, \dots, n-2$. Then for each $i = 1, \dots, n-1$ there is a point $x_i \in \overline{C}$ such that $d(v_i, x_i) \leq 102\delta$. Put $x_0 = x$ and $x_n = y$. Then the sequence $x = x_0, x_1, \dots, x_n = y$ is a required λ-quasigeodesic sequence connecting x and y inside $\bar{C}$ (in fact it is a $(204\delta + 1)$-quasigeodesic).

Now $\bar{C}$ is A-invariant and $\bar{C}/A$ is compact since C/A is bounded and X has compact closed metric balls. Thus Lemma 3.5 implies that A is finitely generated. Pick a finite generating set T for A. Take a point $p \in C$ and define $f\colon G \longrightarrow X$ by $f(g) = gp$. Notice that $f(A) \subset C$ since $p \in C$ and C is A-invariant.

The word metric d_T on A is equivalent to the f-pullback of the restriction of d to $\bar{C}$ by Lemma 3.5. On the other hand the word metric d_S on G is equivalent to the f-pullback of d by [ECHLPT, 3.3.6].

Therefore

$$d_T \sim f_*(d|_{\bar{C}}) = f_*(d)|_A \sim d_S|_A.$$

Thus $d_T \sim d_S|_A$ and so A is quasiconvex in G.

LEMMA 3.6. *Let X be a δ-hyperbolic space ($\delta > 1$) and K be a closed subset of ∂X with at least two elements. Let $C = CK$ and K_1 be the limit set of C in ∂X, that is K_1 is the union of all limits of sequences of points of C converging to infinity.*

Then $K = K_1$.

PROOF. Clearly $K \subset K_1$.

Pick a point $p \in C$ and put $C_1 = C \cup \left(\cup_{a \in K}[p,a)\right)$. Then C_1 is contained in the 10δ-neighbourhood of C since ideal triangles in X are 10δ-thin. Indeed, $p \in (x,y)$ for some $x,y \in K$. Let $a \in K$ be an arbitrary element different from x,y. Then the ideal triangle with sides $[p,x)$, $[p,a)$ and (x,a) is 10δ-thin (here $[p,x) \subset (x,y)$). Thus $[p,a)$ is contained in the 10δ-neighbourhood of $[p,x) \cup (x,a)$ and so in the 10δ-neighbourhood of C.

Suppose $b = \lim_{i \to \infty} x_i$ where $x_i \in C$ and $b \in K_1$. By the previous remark, we may assume that each x_i lies on a geodesic $r_i = [p,a_i)$ for some $a_i \in K$. Since ∂X is compact, after passing to a subsequence, we may assume that $\lim a_i = a$. Clearly $a \in K$ since K is closed. Fix a geodesic ray $r = [p,a)$. We are going to show that $a = b$, and thus that $K_1 = K$. Since $\lim a_i = a$, we conclude that $\lim(r_i, r)_p = \infty$. If $b \neq a$ and $x_i \longrightarrow b$, there is $M > 1, M < \infty$ such that $(x_i, r)_p \leq M$ for each i. Now $d(p,x_i) \longrightarrow \infty$. Therefore we may assume, after passing to a subsequence, that for each $i (x_i, r)_p \leq M$ and $d(p,x_i) > 1000M\delta$. But the last two inequalities imply that $(r_i, r)_p$ is 10δ-close to $(x_i, r)_p$ which contradicts $\lim(r_i, r)_p = \infty$. ∎

Recall (see [Bo2] for details) that a discrete group G of isometries of a n-dimensional hyperbolic space $\mathbb{H}^n$ without parabolics (that is any element of infinite order fixes exactly two points in the boundary S^{n-1}) is termed *geometrically finite* if $\mathrm{Conv}(\Lambda(G))/G$ is compact, where $\Lambda(G)$ is a limit set of G in S^{n-1} and $\mathrm{Conv}(\Lambda(G))$ is a (classical) convex hull of the union of all geodesics with endpoints in $\Lambda(G)$. The following theorem, which is due to G. Swarup [Swa], follows immediately from our new definition of quasiconvexity.

THEOREM 3.7 [SWA]. *Let G be a torsion free geometrically finite group of isometries of $\mathbb{H}^n$ without parabolics. Then G is word hyperbolic and a subgroup A of G is quasiconvex in G if and only if A is geometrically finite.*

PROOF. Take X to be the (classical) convex hull of the limit set of G. As a convex subset of $\mathbb{H}^n$, X is 10-hyperbolic with the metric induced from $\mathbb{H}^n$. The group G is geometrically finite and without parabolics. Thus the action of G on X is properly discontinuous and X/G is compact.

Therefore by Theorem 3.3.6 of [ECHLPT] G is finitely generated and has a Cayley graph which is quasi-isometric to a 10-hyperbolic metric space X, so that G is word hyperbolic. The stated properties of quasiconvex subgroups now follow directly from Proposition 3.4 and the definition of geometrical finiteness.

Indeed, if $A \leq G$ is geometrically finite then $\mathrm{Conv}(\Lambda(A))/A$ is compact and so has bounded diameter $M < \infty$. On the other hand $\Lambda(A) = \partial_X(A) = K$ and $CK \subset \mathrm{Conv}(\Lambda(A))$ is A-invariant. Thus $\mathrm{diam}(CK/A) \leq M$ and A is quasiconvex in G by Proposition 3.4.

Suppose now that $A \leq G$ is quasiconvex in G and $\Lambda(A) = \partial_X(A) = K$ as above. Then $\operatorname{diam}(CK/A) = N < \infty$. It is easy to see that $CK \subset \operatorname{Conv}(\Lambda(A))$ and $\operatorname{Conv}(\Lambda(A))$ is contained in a $1000n$-neighborhood of CK (where $n = \dim \mathbb{H}^n$). This implies that $\operatorname{Conv}(\Lambda(A))/A$ has bounded diameter and so is compact. Thus A is geometrically finite.

We now obtain the analogue of Lemma 2.2 (ii).

LEMMA 3.8. *Suppose G is a word hyperbolic group acting by isometries on a proper δ-hyperbolic space X so that the action is properly discontinuous and cocompact. Let A be an infinite subgroup of G, $K = \partial_X(A)$ and $H = \operatorname{Stab}_G(K) = \{g \in G \mid gK = K\}$. Then*

(1) For any subgroup E of H, containing A, $\partial_X(E) = \partial_X(A)$. In particular $\partial_X(H) = \partial_X(A)$.

(2) If A is normal in a subgroup E of G then $\partial_X(E) = \partial_X(A)$.

PROOF. (1) Let $C = CK$. Notice that K has at least two points so that C is nonempty. Pick a point $p \in C$. It is clear that $K = \partial_X(A) \subset \partial_X(E)$. Since K is E invariant, $C = CK$ is also E-invariant. Thus $Ep \subset C$ and therefore $\partial_X(E)$ is contained in the limit set of C which by Lemma 3.6 is equal to K. Therefore $\partial_X(A) = K = \partial_X(E)$.

(2) If A is normal in E, for any $g \in E$ we have $gAp = Agp$. The limit set of Ap is equal to the limit set of Agp and is equal to K. Thus $gK = K$ and so $A \leq E \leq \operatorname{Stab}_G(K)$. By Part (1) this implies $\partial_X(A) = \partial_X(E) = K$.

Commensurability in hyperbolic groups. It is possible to obtain the following criterion of commensurability for quasiconvex subgroups of hyperbolic groups; this is the promised strengthened form of 2.4.

LEMMA 3.9. *Let G be a word hyperbolic group and let E, H be subgroups of G with $H \leq E$. Let (X, d) be a proper δ-hyperbolic metric space ($\delta > 1$) equipped with properly discontinuous isometric cocompact action of G. Suppose that H is infinite and quasiconvex in G and let $K = \partial_X(H)$. Then the following conditions are equivalent.*

(a) H has finite index in E;

(b) the set $C = CK$ is E-invariant;

(c) E is quasiconvex in G and $\partial_X(E) = \partial_X(H)$;

(d) E is contained in $\operatorname{Stab}_G(K)$ where $\operatorname{Stab}_G(K) = \{g \in G \mid gK = K\}$.

PROOF. Notice that if $p \in X$, $h \in H$ then $Hp = hHp$ and so $K = hK$. Thus $H \leq \operatorname{Stab}_G(K)$. Now the implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (d), (d) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) are clear.

Suppose now that (b) is satisfied. Then the closure (in X) $\bar{C}$ is E-invariant and quasiconvex in X. Thus $\operatorname{diam}(C/E) \leq \operatorname{diam}(C/H) < \infty$ and therefore $\bar{C}/E$ is compact. This implies by Lemma 3.5 that E is finitely generated. Moreover, if Y is a finite generating set for G, Z is a finite generating set for $E, p \in CK$ and $f\colon G \longrightarrow X$ is given by $f(g) = gp$, then again by Lemma 3.5:

$$d_Y \sim f_*(d),\ d_Z \sim (f_*(d|_{\bar{C}})|_E) \text{ and so } d_Z \sim \left(f_*(d|_{\overline{C}})\right)|_E = [f_*(d)]_E \sim (d_Y)|_E.$$

Thus E is quasiconvex in G.

Observe that K is a closed subset of ∂X as it is a limit set of a subgroup and so by Lemma 3.6 the limit set of C is K. We claim that $\partial_X(E) = K$. Clearly $K \subset \partial_X(E)$ since $H \leq E$. On the other hand C is E-invariant, $p \in C$ and $Ep \subset C$. The limit set of E which is, by definition, the limit set of Ep is contained in the limit set of the larger set C, and this limit is K. Thus $\partial_X(E) = K$.

We have shown that (b) implies (c). We will now show that (b) implies (a).

Indeed, suppose Y and Z are as above and S is a finite generating set of H. Let e be an arbitrary element of E. Since $\operatorname{diam}(C/H) \leq M < \infty$ and $f(e) = ep \in C$, there is a point $hp \in C$ for some $h \in H$ such that $d(ep, hp) \leq 2M$. Since f is a λ-quasi-isometry from (G, d_Y) onto (X, d), we conclude that $d_Y(e, h) \leq 2M\lambda + \lambda = M_1$. Thus for any $e \in E$ there is an element $g_e \in G$ with $d(g_e, 1) \leq M_1$ such that $eg_e \in H$. Therefore H has finite index in E. This completes the proof of Lemma 3.9.

COROLLARY 3.10. *Let G be a word hyperbolic acting by isometries properly discontinuously and cocompactly on a proper hyperbolic space X. Let H be an infinite quasiconvex subgroup of G and $K = \partial_X(H)$. Then* $\mathrm{Stab}_G(K) = VN_G(H)$.

PROOF. It is clear that $H \leq VN_G(H) \leq \mathrm{Stab}_G(K)$. On the other hand H has finite index in $\mathrm{Stab}_G(K)$ by Lemma 3.9(d). Thus for any $g \in \mathrm{Stab}_G(K)$ we have $|H{:}gHg^{-1} \cap H| \leq \infty$ and $|gHg^{-1} : gHg^{-1} \cap H| \leq \infty$. Therefore $g \in VN_G(H)$ and so $VN_G(H) = \mathrm{Stab}_G(K)$.

Theorem 1 now follows immediately from this proposition, as in Section 2.

PROOF OF THEOREM 1. Let X be the Cayley graph of G with respect to some finite generating set S. Put $K = \partial_X(A)$.

(1) It is obvious that $\partial_X(A) = \partial_X(B) = \partial_X(A \cap B)$. Denote this set by K. Thus $A, B \leq \mathrm{Stab}_G(K)$ and so $E = sgp(A \cup B) \leq \mathrm{Stab}_G(K)$. Thus by Lemma 3.9(d) $A \cap B$ has finite index in E.

(2) Suppose there are infinitely many subgroups of G containing A as a subgroup of finite index. Let E be the subgroup of G generated by their union. Then A has infinite index in E. On the other hand any subgroup of G having A as a subgroup of finite index has K as its limit set and so leaves K invariant. Thus K is E-invariant, that is $E \leq \mathrm{Stab}_G(K)$. Therefore A has finite index in E by Lemma 3.9(d). This contradicts our assumptions.

(3) It is clear that $A \leq VN_G(A)$. Suppose now that $g \in VN_G(A)$. Put $U = gAg^{-1} \cap H$. Then U has finite index in A and in gAg^{-1}. Since U is of finite index in A and in gAg^{-1}, $K = \partial_X(U) = \partial_X(A) = \partial_X(gAg^{-1})$. However $\partial_X(gAg^{-1})$ is equal to the limit set of gA that is to gK. Thus $gK = K$ and $g \in \mathrm{Stab}_G(K)$. Since $g \in VN_G(A)$ was chosen arbitrarily, we conclude that $A \leq VN_G(A) \leq \mathrm{Stab}_G(K)$. Thus by Lemma 3.9 A has finite index in $VN_G(A)$.

PROOF OF THEOREM 3. Let X be the Cayley graph of G with respect to some finite generating set S. Put $K = \partial_X(A)$. The "if" implication is clear. So suppose that A contains an infinite subgroup C which is normal in B. Then by Lemma 3.8 $\partial_X(C) = \partial_X(B)$. Since $C \leq A \leq B$, we conclude that $K = \partial_X(A) = \partial_X(B)$. Thus $B \leq \mathrm{Stab}_G(K)$ and by Lemma 3.9(d) A has finite index in B.

4. Some additional remarks about the elementary boundary. Let S be a finite generating set for the word hyperbolic group G and (X, d_S) be the Cayley graph of G with respect to S. Fix a number $\delta > 1$ such that (X, d_S) is δ-hyperbolic. The elementary boundary $\tilde{\partial} G$ (in the sense of Section 2) is a subset of ∂X (in the sense of Section 3) and moreover, a dense subset (see [Gr]). The map from the elementary boundary into the Gromov boundary is given by $[hg^+] \longmapsto \lim_{n\to\infty} hg^n h^{-1} = \lim_{n\to\infty} hg^n$.

It is also not hard to see, using the compactness argument, that if $g \in \mathcal{H}_G(Y)$ for a subset Y of $\tilde{\partial} G$ then there is a pair of points $a(g), a(h) \in Y$ such that g lies on a certain geodesic $\big(a(g), a(h)\big)$ (this is a bi-infinite geodesic the sense of Section 3). Moreover, if $a(g), a(h) \in Y$ are two distinct points, then there is a geodesic $\big(a(g), a(h)\big)$ such that any point (vertex) of $\big(a(g), a(h)\big)$ belongs to $\mathcal{H}_G(Y)$.

This immediately implies

LEMMA 4.1. *Let G, X, δ be as above, and let Y be a subset of $\tilde{\partial} G$ with at least two elements.*

Then $\mathcal{H}_G(Y)$ is 100δ-quasiconvex in X.

PROOF. The proof is exactly the same as that of Lemma 3.2 if one takes into account the remarks above.

The previous observation combined with Corollary 2.4 allows one to obtain a criterion of quasiconvexity similar to Proposition 3.4 in terms of elementary boundary, making the condition of 2.4 necessary and sufficient.

PROPOSITION 4.2. *Let G be a word hyperbolic group, S a finite generating set and X be the Cayley graph of G with respect to S. Let A be an infinite subgroup of G.*

The following conditions are equivalent:

(i) A is quasiconvex in G;

(ii) the quotient $\mathcal{H}(\tilde{\partial}_G A)/A$ is finite.

PROOF. The implication (i) $\Rightarrow$ (ii) was established in Corollary 2.4. Suppose now that (ii) is satisfied. As A is infinite, $\tilde{\partial} A$ contains at least two points and so $C = \mathcal{H}(\tilde{\partial}_G A)$ is nonempty and even infinite. Now C is 30δ-quasiconvex in X by Lemma 4.1 and so it is λ-quasigeodesic for $\lambda = 3000\delta$ (the argument is exactly the same as in the proof of Proposition 3.4). Now C is discrete (as $C \subset G$) and has compact metric balls. Fix a basepoint $p \in C$. We know that C/A is finite and therefore compact. Thus by Lemma 3.5, A is finitely generated and the orbit map $a \longmapsto ap, a \in A$ is a quasi-isometry of A onto C. The rest of the argument goes exactly as in Proposition 3.4.

REFERENCES

[ABC] J. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro and H. Short, *Notes on hyperbolic groups*, In: Group theory from a geometric viewpoint, Proc. ICTP. Trieste, World Scientific, Singapore, 1991, 3–63.

[Ba] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89**(1993), 3–47.

[Bo1] B. Bowditch, *Notes on Gromov's hyperbolicity criterion for path-metric spaces*, In: Group theory from a geometric viewpoint, (eds. E. Ghys, A. Haefliger and A. Verkovsky), 1991, 64–167.

[Bo2] B. Bowdich, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. (2)**113**(1993), 245–317.

[Br] N. Brady, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, University of Utah, 1995, preprint.

[BMS] G. Baumslag, C. F. Miller III and H. Short, *Unsolvable problems about small cancellation and word hyperbolic groups*, Bull. London Math. Soc. (1)**26**(1994), 97–101.

[BS] M. Bridson and G. Swarup, *On Hausdorff-Gromov convergence and a theorem of Paulin*, Enseign. Math. (2)**40**(1994), no. 3–4, 267–289.

[CDP] M. Coornaert, T. Delzant and A. Papadopoulos, *Notes sur les groupes hyerboliques de Gromov*, no. 1441, Springer-Verlag, 1990.

[De] M. Dehn, *Papers on Group Theory and Topology*, translated and introduced by J. Stillwell, Springer-Verlag, 1987.

[ECHLPT] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett, MA, 1992.

[F] W. Floyd, *Group completions and limit sets of Kleinian groups*, Invent. Math. **57**(1980), 205–218.

[G] L. Greenberg, *Discrete Groups of Motions*, Canad. J. Math. **12**(1960), 414–425.

[GH] E. Ghys and P. de la Harpe (editors), *Sur les groupes hyperboliques d'après Mikhael Gromov*, In: Papers from the Swiss seminar on hyperbolic groups held in Bern, 1988, (eds. E. Ghys and P. de la Harpe), Progress in Mathematics **83**, 1990.

[Gr] M. Gromov, *Hyperbolic Groups*, In: Essays in group theory, (ed. S. M. Gersten), MSRI Publ. **8**, Springer, 1987, 75–263.

[GMRS] R. Gitik, M. Mitra, E. Rips and M. Sageev, *Widths of subgroups*, Trans. Amer. Math. Soc., to appear.

[GS] S. M. Gersten and H. Short, *Rational subgroups of biautomatic groups*, Ann. of Math. (2)**134**(1991), no. 1, 125–158.

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin-Heidleberg-New York, 1977.

[Mo] J. Morgan, *On Thurston's uniformization theorem for three-dimensional manifolds*, In: The Smith Conjecture, (eds. H. Bass and J. Morgan), Academic Press, 1984, 37–126.

[MT] M. Mihalik and W. Towle, *Quasiconvex subgroups of negatively curved groups*, J. Pure Appl. Algebra **3**(1995), 297–301.

[N] W. Neumann, *The fixed group of an automorphism of a word hyperbolic group is rational*, Invent. Math. **110**(1992), 147–150.

[Pi] Ch. Pittet, *Surface groups and quasiconvexity*, In: Geometric Group Theory **1**, Sussex, 1991, London Math. Soc. Lecture Notes Series **181**, Cambridge University Press, Cambridge, 1993, 169–175.

[R] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math Soc. **14**(1982), 45–47.

[Sho] H. Short, *Quasiconvexity and a Theorem of Howson's*, In: Group theory from a geometric viewpoint, Proc. ICTP. Trieste, World Scientific, Singapore, 1991.

[Sta] J. R. Stallings, *Topology of finite graphs*, Invent. Math. **71**(1983), 551–565.

[Swa] G. A. Swarup, *Geometric finiteness and rationality*, J. Pure Appl. Algebra **86**(1993), 327–333.

[Swe] E. Swensen, *Limit sets in the boundary of word hyperbolic groups*, 1994, preprint.

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THE RESIDUAL SPECTRUM OF G_2

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ABSTRACT. We completely determine the residual spectrum of the split exceptional group of type G_2, thus completing the work of Langlands and Moeglin-Waldspurger on the part of the residual spectrum attached to the trivial character of the maximal torus. We also give the Arthur parameters for the residual spectrum coming from Borel subgroups. The interpretation in terms of Arthur parameters explains the "bizarre" multiplicity condition in Moeglin-Waldspurger's work. It is related to the fact that the component group of the Arthur parameter is non-abelian.

1. **Introduction.** Let F be a number field and $(\mathbb{A})$ its ring of adeles. Let G be a reductive group. A central problem in the theory of automorphic forms is to decompose the right regular representation of $G(\mathbb{A})$ acting on the Hilbert space $L^2\big(G(F)\backslash G(\mathbb{A})\big)$. It has the continuous spectrum and the discrete spectrum:

$$L^2\big(G(F)\backslash G(\mathbb{A})\big) = L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big) \oplus L^2_{\text{cont}}\big(G(F)\backslash G(\mathbb{A})\big).$$

We are mainly interested in the discrete spectrum. Langlands [L4] described, using his theory of Eisenstein series, an orthogonal decomposition:

$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big) = \bigoplus_{(M,\pi)} L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)},$$

where (M,π) is a Levi subgroup with a cuspidal automorphic representation π taken modulo conjugacy. (Here we normalize π so that the action of the maximal split torus in the center of G at the archimedean places is trivial.) $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)}$ is a space of residues of Eisenstein series associated to (M,π). Here we note that the subspace

$$\bigoplus_{(G,\pi)} L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(G,\pi)},$$

is the space of cuspidal representations $L^2_{\text{cusp}}\big(G(F)\backslash G(\mathbb{A})\big)$. Its orthogonal complement in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ is called the residual spectrum and we denote it by $L^2_{\text{res}}\big(G(F)\backslash G(\mathbb{A})\big)$. Therefore we have an orthogonal decomposition

$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A}) = L^2_{\text{cusp}}\big(G(F)\backslash G(\mathbb{A})\big) \oplus L^2_{\text{res}}\big(G(F)\backslash G(\mathbb{A})\big).$$

Arthur described a conjectural decomposition of this space as follows:

$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big) = \bigoplus_{\psi} L^2\big(G(F)\backslash G(\mathbb{A})\big)_{\psi},$$

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where ψ runs, modulo conjugacy, through the set of morphism:

$$\psi: L_F \times \mathrm{SL}(2,\mathbb{C}) \longmapsto G^*,$$

where L_F is the conjectural tannakian group, G^* is the Langlands' L-group and ψ satisfies certain conditions; in particular, ψ restricted to $\mathrm{SL}(2,\mathbb{C})$ is algebraic, ψ restricted to L_F parametrize a cuspidal tempered representation of a Levi subgroup and the image of ψ is not included in proper Levi subgroups. The space $L^2\big(G(F)\backslash G(\mathbb{A})\big)_\psi$ is defined by local data (See Section 3.7).

The purpose of this paper is to determine explicitly $L^2_{\mathrm{res}}\big(G(F)\backslash G(\mathbb{A})\big)$ for G the split exceptional group of type G_2. There are 3 Levi subgroups modulo conjugacy. Let M_1 be the Levi of the maximal parabolic subgroup P_1 attached to the long simple root and M_2 be the Levi of the maximal parabolic subgroup P_2 attached to the short simple root. Let $M_0 = T$ be the maximal split torus. Let

$$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i} = \bigoplus L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(M_i,\pi)}.$$

Theorems 3.6.1, Theorem 4.2 and Theorem 5.1 describe the decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i}$ for $i = 0, 1, 2$, respectively. Due to the lack of information on the poles of the adjoint cube L-function of GL_2 (symmetric cube L-function of GL_2 twisted by a character) and insufficient bound for Fourier coefficients of cuspidal representations of GL_2, our results on $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i}$, $i = 1, 2$, are incomplete. Also we only consider the K-finite, K_∞-invariant subspace of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. We also give the Arthur parameter for automorphic representations in $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$ and verify Arthur's conjecture, reformulated by Moeglin [M3].

In order to obtain the decomposition for $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$, we use the inner product formula (3.1) of the pseudo-Eisenstein series as in [M-W1]. For $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_i}$, $i = 1, 2$, we use the method in [Ki], where we calculated the residual spectrum of Sp_4. We use the notation of [M-W1]. Let us explain in detail in each of the cases.

The most interesting one is the analysis of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. It was Langlands [L4, Appendix 3] who first calculated its K-fixed subspace. It is dimension 2. One is spherical and the other is a very interesting residual automorphic representation. Its Archimedean component is infinite dimensional, of class one and is not tempered. Moeglin and Waldspurger [M-W1, Appendix III] calculated K-finite, K_∞-invariant subspace V of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,1)}$, where 1 is the trivial character of T, whose cuspidal exponents are short roots. They found surprising results that only those which satisfy a certain condition appear in $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. Let us explain in detail. Let $J_v = \{\pi_{1v}, \pi_{2v}\}$ be a set of irreducible representations of G_v, where π_{1v} is spherical. For S a finite set of finite places, set $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$. Then

$$V = \bigoplus_{S,\mathrm{card}(S)\neq 1} \pi^S.$$

The condition $\mathrm{card}(S) \neq 1$ is quite surprising (In Sp_4 case [Ki], we have the condition "$\mathrm{card}(S)$ even"). We can interpret their results in terms of Arthur parameters. In fact, the condition comes from the Springer correspondence (see Section 3.7 for more details) and is related to the fact that $A(u)$ is non-abelian, *i.e.*, S_3, the symmetric group on 3 letters. Recall that the Springer correspondence is an injective map from the set of irreducible characters of W, the Weyl group of G, into the set of pairs (O, η), where O is a unipotent orbit and η is an irreducible character of $A(u) = C(u)/C(u)^0$, $u \in O$ and $C(u)$ is the centralizer of u. Let Springer(O) be the set of characters of $A(u)$ which are in the image of Springer correspondence. Then J_v is associated with $\text{Springer}\big(G_2(a_1)\big)$, where $G_2(a_1)$ is the sub-regular unipotent orbit of G_2 ([Ca, p.401]). We note that Moeglin [M1] proved that the residual spectrum attached to the trivial character of the torus is parametrized by distinguished unipotent orbits O and Springer(O) and we can expect that the same thing would happen for all split groups.

Among non-trivial characters of $T(\mathbb{A})/T(F)$, modulo conjugacy, the following characters of the torus contribute to the residual spectrum (see Section 3.6). Under the identification, $M_1 \simeq \mathrm{GL}_2$, where M_1 is the Levi subgroup of P_1, $\chi = \chi(\mu, \nu)$, μ and ν are grõssencharacters of F.

(1) $\mu = \nu, \mu^2 = 1, \mu \neq 1$

(2) $\mu^3 = 1, \mu \neq 1, \nu = \mu^2$.

For Case (2), there is only one residual spectrum, which is the global Langlands' quotient of $\mathrm{Ind}_{P_1}^{G} \exp\big(\beta_4, H_{P_1}()\big) \otimes (\mathrm{Ind}_{B_0}^{\mathrm{GL}_2} \chi)$. Case (1) is more interesting. In this case the Eisenstein series has a pole at β_2, the sum of two simple positive roots. If μ_v is not trivial, then the character $\chi_v \otimes \exp\big(\beta_2, H_B(\ )\big)$ is regular and we can apply Rodier's result ([R]) to analyze the image of the intertwining operator $R(\rho_2, \beta_2, \rho_2\chi)$. In particular, it is irreducible. If μ_v is trivial, then the image of $R(\rho_2, \beta_2, \rho_2\chi)$ is the same as the one Moeglin and Waldspurger found [M-W1, Appendix III]. It is the sum of two irreducible representations. Let $J(\chi_v) = \{\pi_{1v}, \pi_{2v}\}$ is the set of irreducible components. We put $\pi_{2v} = 0$ if $\chi_v \neq 1$. For S a finite set of finite places, set $\pi^S = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$. Then the residual spectrum attached to the character (1) is given by

$$J(\chi) = \bigoplus_S \pi^S.$$

There is no condition on S.

For $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$, the Levi factor is $M_1 = \mathrm{GL}_2$. We have to look at Eisenstein series associated to cuspidal representations of GL_2. The L-functions in the constant terms of Eisenstein series are the adjoint cube L-function and Hecke L-function. We analyze the poles and the irreducibility of the images of local intertwining operators as in [Ki]. However, the pole of the adjoint cube L-function of GL_2 is not known. We assume the location of poles. Also we need to assume certain estimates of Fourier coefficients of cuspidal representations of GL_2. More precisely, if $\pi_v = \pi(\mu|\ |^r, \mu|\ |^{-r})$ is a complementary series representation of GL_2, we assume that $r < \frac{1}{6}$. Right now the best known result is that $r < \frac{1}{5}$ due to Shahidi [S3]. Assuming these facts, we obtain a decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$, parametrized by cuspidal representations π of GL_2 with trivial central characters and $L(\frac{1}{2}, \pi, r_3^0) \neq 0$ and by monomial representations of GL_2. Recall that a cuspidal representation σ of GL_2 is called monomial if $\sigma \simeq \sigma \otimes \eta$ for a quadratic grössencharacter η of F.

For $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2}$, the Levi factor is $M_2 = \mathrm{GL}_2$. The L-functions in the constant term of the Eisenstein series are just Jacquet-Langlands' L-function, its twist by a character, and a Hecke L-function. In this case, assuming the fact on the estimates of Fourier coefficients of cuspidal representations of GL_2, we obtain a decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2}$, parametrized by cuspidal representations π of GL_2 with trivial central characters and $L(\frac{1}{2}, \pi) \neq 0$.

F. Shahidi brought to our attention the paper by Li and Schwermer [Li-Sc] who studied the poles of Eisenstein series attached to the maximal parabolic subgroups over $\mathbb{Q}$. After this paper was accepted, the author received a preprint, "The residual spectrum of the group of type G_2," by S. Zampera. She obtained similar results. But her result does not have the interpretation in terms of Arthur parameters.

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2. **Some facts about G_2; roots and parabolic subgroups.** Let G be a split group of type G_2. Fix a Cartan subgroup T in G and let $B = TU$ be a Borel subgroup of G. Let K_∞ be the standard maximal compact subgroup in $G(\mathbb{A}_\infty)$ and $K_v = G(O_v)$ for finite v. The product $K = K_\infty \times \prod K_v$ is a maximal compact subgroup in $G(\mathbb{A})$.

We follow Moeglin and Waldspurger [M-W1, Appendix III]. Let β_1 be a long simple root and β_6 a short one. Let

$$\beta_2 = \beta_1 + \beta_6, \quad \beta_3 = 2\beta_1 + 3\beta_6, \quad \beta_4 = \beta_1 + 2\beta_6, \quad \beta_5 = \beta_1 + 3\beta_6$$

then $\{\beta_1, \ldots, \beta_6\}$ is a set of positive roots.

Let $\beta_i^\vee$ be a corresponding coroot of β_i for $i = 1, \ldots, 6$. Then

$$\beta_2^\vee = 3\beta_1^\vee + \beta_6^\vee, \quad \beta_3^\vee = 2\beta_1^\vee + \beta_6^\vee, \quad \beta_4^\vee = 3\beta_1^\vee + 2\beta_6^\vee, \quad \beta_5^\vee = \beta_1^\vee + \beta_6^\vee.$$

Let P_1 the maximal parabolic subgroup generated by β_1 (long root) and P_2 be the maximal parabolic subgroup generated by β_6 (short root). Then we have Levi decompositions (Shahidi [S4]):

$$P_1 = M_1 N_1, \quad P_2 = M_2 N_2, \quad M_1 \simeq \mathrm{GL}_2, \quad M_2 \simeq \mathrm{GL}_2.$$

Under the identification $M_1 \simeq \mathrm{GL}_2$,

$$\begin{aligned} &\beta_1^\vee(t) = \mathrm{diag}(t, t^{-1}),\ \beta_6^\vee(t) = \mathrm{diag}(t^{-1}, t^2),\ \beta_2^\vee(t) = \mathrm{diag}(t^2, t^{-1}) \\ &\beta_3^\vee(t) = \mathrm{diag}(t, 1),\ \beta_4^\vee(t) = \mathrm{diag}(t, t),\ \beta_5^\vee(t) = \mathrm{diag}(1, t). \end{aligned} \tag{2.1}$$

Under the identification $M_2 \simeq \mathrm{GL}_2$,

$$\beta_1^\vee(t) = \mathrm{diag}(1,t),\ \beta_6^\vee(t) = \mathrm{diag}(t,t^{-1}),\ \beta_2^\vee(t) = \mathrm{diag}(t,t^2)$$
$$\beta_3^\vee(t) = \mathrm{diag}(t,t),\ \beta_4^\vee(t) = \mathrm{diag}(t^2,t),\ \beta_5^\vee(t) = \mathrm{diag}(t,1). \tag{2.2}$$

Let $X(T)$ (resp. $X^*(T)$) be the character (resp. cocharacter) group of T. Since G is simply connected,

$$X(T) = \mathbf{Z}\beta_3 + \mathbf{Z}\beta_4, \quad X^*(T) = \mathbf{Z}\beta_1^\vee + \mathbf{Z}\beta_6^\vee.$$

Let

$$\mathfrak{a}^* = X(T)\otimes\mathbf{R}, \quad \mathfrak{a}_{\mathbf{C}}^* = X(T)\otimes\mathbf{C}, \quad \mathfrak{a} = X^*(T)\otimes\mathbf{R} = \mathrm{Hom}\big(X(T),\mathbf{R}\big)$$
$$\mathfrak{a}_{\mathbf{C}} = X^*(T)\otimes\mathbf{C}.$$

Then $\{\beta_3,\beta_4\}$ and $\{\beta_1^\vee,\beta_6^\vee\}$ are the pair of dual bases for $\mathfrak{a}^*$ and $\mathfrak{a}$, respectively.

The positive Weyl Chamber in $\mathfrak{a}^*$ is

$$\begin{aligned} C^+ &= \{\Lambda \in \mathfrak{a}^* \mid \langle \Lambda, \alpha^\vee\rangle > 0 \text{ for all } \alpha \text{ positive roots }\} \\ &= \{a\beta_3 + b\beta_4 \mid a,b > 0\}. \end{aligned}$$

The half sum of positive roots is $\rho_B = \beta_3 + \beta_4$.

We list the elements of the Weyl group together with their actions on the positive roots.

w	decomposition	β_1	β_2	β_3	β_4	β_5	β_6
1		β_1	β_2	β_3	β_4	β_5	β_6
ρ_1	ρ_1	$-\beta_1$	β_6	β_5	β_4	β_3	β_2
ρ_2	$\rho_1\rho_6\rho_1$	$-\beta_3$	$-\beta_2$	$-\beta_1$	β_6	β_5	β_4
ρ_3	$\rho_1\rho_6\rho_1\rho_6\rho_1$	$-\beta_5$	$-\beta_4$	$-\beta_3$	$-\beta_2$	$-\beta_1$	β_6
ρ_4	$\rho_6\rho_1\rho_6\rho_1\rho_6$	β_1	$-\beta_6$	$-\beta_5$	$-\beta_4$	$-\beta_3$	$-\beta_2$
ρ_5	$\rho_6\rho_1\rho_6$	β_3	β_2	β_1	$-\beta_6$	$-\beta_5$	$-\beta_4$
ρ_6	ρ_6	β_5	β_4	β_3	β_2	β_1	$-\beta_6$
$\sigma(\frac{\pi}{3})$	$\rho_6\rho_1$	$-\beta_5$	$-\beta_6$	β_1	β_2	β_3	β_4
$\sigma(\frac{2\pi}{3})$	$\rho_6\rho_1\rho_6\rho_1$	$-\beta_3$	$-\beta_4$	$-\beta_5$	$-\beta_6$	β_1	β_2
$\sigma(\pi)$	$\rho_6\rho_1\rho_6\rho_1\rho_6\rho_1$	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$-\beta_5$	$-\beta_6$
$\sigma(\frac{4\pi}{3})$	$\rho_1\rho_6\rho_1\rho_6$	β_5	β_6	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$
$\sigma(\frac{5\pi}{3})$	$\rho_1\rho_6$	β_3	β_4	β_5	β_6	$-\beta_1$	$-\beta_2$

3. **Decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$.**

We fix an additive character $\psi = \otimes_v \psi_v$ of $\mathbb{A}/F$ and let $\xi(z,\mu)$ be the Hecke L-function with the ordinary Γ-factor so that it satisfies the functional equation $\xi(z,\mu) = \epsilon(z,\mu)\xi(1-z,\mu^{-1})$, where $\epsilon(z,\mu) = \prod_v \epsilon(z,\mu_v,\psi_v)$ is the usual ϵ-factor (see, for example, [Go, p158]). If μ is the trivial character μ_0, then we write simply $\xi(z)$ for $\xi(z, \mu_0)$. We have the Laurent expansion of $\xi(z)$ at $z = 1$:

$$\xi(z) = \frac{c(F)}{z-1} + a + \cdots.$$

3.1. *Definition of Eisenstein Series.* For a unitary character χ of $T(\mathbb{A})/T(F)$ and for each $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, let $I(\Lambda, \chi) = \mathrm{Ind}_B^G \chi \otimes \exp\big(\Lambda, H_B(\ )\big)$ be the induced representation, where H_B is the homomorphism $H_B\colon T(\mathbb{A}) \longmapsto \mathfrak{a}$ defined by

$$\exp\langle \chi, H_B(t)\rangle = \prod_v |\chi_v(t_v)|_v.$$

We form the Eisenstein series:

$$E(g,f,\Lambda) = \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g),$$

where $f \in I(\Lambda, \chi)$. The Eisenstein series converges absolutely for $\mathrm{Re}\,\Lambda \in C^+ + \rho_B$ and extends to a meromorphic function of Λ. It is an automorphic form and the constant term of $E(g,f,\Lambda)$ along B is given by

$$E_0(g,f,\Lambda) = \int_{U(F)\backslash U(\mathbb{A})} E(ug,f,\Lambda)\,du = \sum_{w \in W} M(w,\Lambda,\chi)f(g),$$

where W is the Weyl group and for sufficiently regular Λ,

$$M(w,\Lambda,\chi)f(g) = \int_{U_w(\mathbb{A})} f(w^{-1}ug)\,du,$$

where $U_w = U \cap w\bar{U}w^{-1}$, $\bar{U}$ is the unipotent radical opposed to U. Then $M(w,\Lambda,\chi)$ defines a linear map from $I(\Lambda,\chi)$ to $I(w\Lambda, w\chi)$ and satisfies the functional equation of the form

$$M(w_1w_2,\Lambda,\chi) = M(w_1, w_2\Lambda, w_2\chi)M(w_2,\Lambda,\chi).$$

The Eisenstein series satisfies the functional equation

$$E\big(g, M(w,\Lambda,\chi)f, w\Lambda\big) = E(g,f,\Lambda).$$

Let S be a finite set of places of F, including all the archimedean places such that for every $v \notin S$, χ_v and ψ_v are unramified and if $f = \otimes f_v$, for $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. We have

$$M(w,\Lambda,\chi) = \bigotimes_v M(w,\Lambda,\chi_v).$$

Then by applying Gindikin-Karpelevic method (Langlands [L4]), we can see that for $v \notin S$,

$$M(w,\Lambda,\chi_v)f_v = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle \Lambda, \alpha^\vee\rangle, \chi_v \circ \alpha^\vee)}{L(\langle \Lambda, \alpha^\vee\rangle + 1, \chi_v \circ \alpha^\vee)}\tilde{f}_v,$$

where $L(s,\eta_v)$ is the local Hecke L-function attached to a character η_v of $F_v^\times$ and $s \in \mathbb{C}$ and $\tilde{f}_v$ is the K_v-fixed function in the space of $I(w\Lambda, w\chi)$ satisfying $\tilde{f}_v(e_v) = 1$. For any v, let

$$r_v(w) = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)}{L(\langle\Lambda,\alpha^\vee\rangle+1, \chi_v\circ\alpha^\vee)\epsilon(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee, \psi_v)}.$$

We normalize the intertwining operator $M(w,\Lambda,\chi_v)$ for all v by

$$M(w,\Lambda,\chi_v) = r_v(w)R(w,\Lambda,\chi_v).$$

Let $R(w,\Lambda,\chi) = \otimes_v R(w,\Lambda,\chi_v)$ and $R(w,\Lambda,\chi)$ satisfies the functional equation

$$R(w_1w_2,\Lambda,\chi) = R(w_1, w_2\Lambda, w_2\chi)R(w_2,\Lambda,\chi).$$

We know, by Winarsky [Wi] for p-adic cases and Shahidi [S2, p110] for real and complex cases, that

$$M(w,\Lambda,\chi_v)\prod_{\alpha>0,w\alpha<0} L(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)^{-1}$$

is holomorphic for any v. So for any v, $R(w,\Lambda,\chi_v)$ is holomorphic for Λ with $\mathrm{Re}(\langle\Lambda,\alpha^\vee\rangle) > -1$, for all positive α with $w\alpha < 0$.

We note that a character χ of $T(F)\backslash T(\mathbb{A})$ defines a cuspidal representation of T. For any $w \in W$, $wTw^{-1} = T$ and so $(T, w\chi)$ is conjugate to (T,χ).

Let $I(\chi)$ be the set of entire functions f of Paley-Wiener type such that $f(\Lambda) \in I(\Lambda,\chi)$ for each Λ. Let

$$\theta_f(g) = \left(\frac{1}{2\pi i}\right)^2 \int_{\mathrm{Re}\,\Lambda=\Lambda_0} E\big(g, f(\Lambda), \Lambda\big)\, d\Lambda,$$

where $\Lambda_0 \in \rho_B + C^+$. Then we have

LEMMA ([L4]). *$L^2\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$ spanned by θ_f for all $f \in I(w\chi)$ as $w\chi$ runs through all distinct conjugates of χ.*

$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$ is the discrete part of $L^2\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$. It is the set of iterated residues of $E\big(g, f(\Lambda), \Lambda\big)$ of order 2.

In order to decompose $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$, we use the inner product formula of two pseudo-Eisenstein series: Let χ and χ' be conjugate characters and $f \in I(\chi)$, $f' \in I(\chi')$. Then

$$\langle\theta_f, \theta_{f'}\rangle = \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}\,\Lambda=\Lambda_0} \sum_{w\in W(\chi,\chi')} \big(M(w,\Lambda)f(\Lambda), f'(-w\bar{\Lambda})\big)\, d\Lambda,$$

where $W(\chi,\chi') = \{w \in W \mid w\chi = \chi'\}$. Let $\{d\chi \mid d \in D\}$ be the set of distinct conjugates of χ.

In order to deal with the distinct conjugates of χ simultaneously, we consider, for $f \in I(\chi)$,

$$A(f,f';\Lambda) = \sum_{d\in D}\sum_{w\in W(\chi,d\chi)} \big(M(w,\Lambda,\chi)f(\Lambda), f'_d(-w\bar{\Lambda})\big),$$

where $f'_d \in I(d\chi)$. Since $W = \bigcup_{d\in D} W(\chi, d\chi)$, for simplicity, we write it as

$$A(f,f';\Lambda) = \sum_{w\in W} \big(M(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big). \tag{3.1}$$

We also have the adjoint formula for the intertwining operators

$$\begin{aligned} \big(M(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big) &= \big(f(\Lambda), M(w^{-1}, -w\bar{\Lambda}, w\chi)f'(-w\bar{\Lambda})\big) \\ \big(R(w,\Lambda,\chi)f(\Lambda), f'(-w\bar{\Lambda})\big) &= \big(f(\Lambda), R(w^{-1}, -w\bar{\Lambda}, w\chi)f'(-w\bar{\Lambda})\big). \end{aligned} \tag{3.2}$$

We use this adjoint formula and calculate the residue of $A(f,f';\Lambda)$ to obtain the residual spectrum $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$.

Let $S_i = \{\Lambda \in \mathfrak{a}^*_{\mathbb{C}} : \langle\Lambda, \beta_i^\vee\rangle = 1\}$ and we introduce a coordinate on S_i as follows: $\Lambda = zu_i + \frac{\beta_i}{2}$, where $u_1 = \beta_4, u_2 = \beta_5, u_3 = \beta_6, u_4 = \beta_1, u_5 = \beta_2, u_6 = \beta_3$. We note that $\langle u_i, \beta_i^\vee\rangle = 0$.

In order to get discrete spectrum, we have to deform the contour $\operatorname{Re}\Lambda = \Lambda_0$ to $\operatorname{Re}\Lambda = 0$. Since the poles of the functions $M(w,\Lambda,\chi)$ all lie on S_i which is defined by real equations, we can represent the process of deforming the contour and the singular hyperplanes S_i as dashed lines by the following diagram in the real plane as in [L4, Appendix 3].

The integral at $\operatorname{Re}\Lambda = 0$,

$$\frac{1}{(2\pi i)^2}\int_{\operatorname{Re}\Lambda=0} A(f,f';\Lambda)\,d\Lambda$$

gives the continuous spectrum of dimension 2. As can be seen in the diagram, if we move the contour along the dotted line indicated we may pick up residues at the points $\lambda_i, i = 1,\ldots,6$:

$$\frac{1}{2\pi i}\int_{\operatorname{Re}\Lambda=\lambda_i} \operatorname{Res}_{S_i} A(f,f';\Lambda)\,d\Lambda,$$

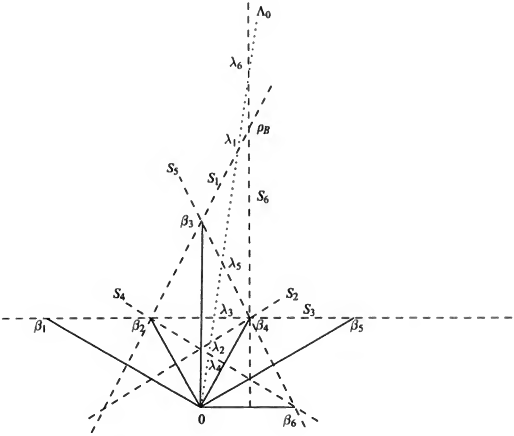
where $\Lambda \in S_i$. Then we deform the contours $\operatorname{Re}\Lambda = \lambda_i$ to $\operatorname{Re}\Lambda = \frac{\beta_i}{2}$, *i.e.*, the origin of S_i. The integrals at $\operatorname{Re}\Lambda = \frac{\beta_i}{2}$,

$$\frac{1}{2\pi i}\int_{\operatorname{Re}\Lambda=\frac{\beta_i}{2}} \operatorname{Res}_{S_i} A(f,f';\Lambda)\,d\Lambda$$

give the continuous spectrum of dimension 1. The square integrable residues which arise during the deformation span the discrete spectrum.

As we see in the diagram, we have to consider

$\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda)$
$\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda)$



$\mathrm{Res}_{\frac{1}{3}\beta_3}\,\mathrm{Res}_{S_2}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\beta_4}\,\mathrm{Res}_{S_5}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\rho_B}\,\mathrm{Res}_{S_6}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\beta_4}\,\mathrm{Res}_{S_6}\,A(f,f';\Lambda)$
$\mathrm{Res}_{\frac{1}{3}\beta_5}\,\mathrm{Res}_{S_6}\,A(f,f';\Lambda)$.

For χ a character of $T(\mathbb{A})/T(F)$, let $\chi_i = \chi \circ \beta_i^\vee$.
Then

$$\chi_2 = \chi_1^3\chi_6, \chi_3 = \chi_1^2\chi_6, \chi_4 = \chi_1^3\chi_6^2, \chi_5 = \chi_1\chi_6.$$

Set

$$M^i(w,\Lambda,\chi) = \frac{\xi(2)}{c(F)}\,\mathrm{Res}_{S_i}\,M(w,\Lambda,\chi).$$

Let $W_i = \{w \in W \mid w\beta_i < 0\}$ for $i = 1,\ldots,6$. Then

$$\mathrm{Res}_{S_i}\,A(f,f',\Lambda) = \sum_{w\in W_i}\Big(M^i(w,\Lambda,\chi)f(\Lambda),f'(-w\bar{\Lambda})\Big).$$

3.2. *Calculation of* $\mathrm{Res}_{S_1}\,A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_1 if $\chi_1 = 1$.
On S_1,

$$\langle \Lambda, \beta_2^\vee \rangle = z + \frac{3}{2}, \langle \Lambda, \beta_3^\vee \rangle = z + \frac{1}{2}.$$

$$\langle \Lambda, \beta_4^\vee \rangle = 2z, \langle \Lambda, \beta_5^\vee \rangle = z - \frac{1}{2}, \langle \Lambda, \beta_6^\vee \rangle = z - \frac{3}{2}.$$

LEMMA 3.2.1.

$$M^1(\rho_1, \Lambda, \chi) = R(\rho_1, \Lambda, \chi)$$

$$M^1(\rho_2, \Lambda, \chi) = \frac{\xi(z+\frac{1}{2}, \chi_6)\, R(\rho_2, \Lambda, \chi)}{\xi(z+\frac{5}{2}, \chi_6)\varepsilon(z+\frac{1}{2}, \chi_6)\varepsilon(z+\frac{3}{2}, \chi_6)}$$

$$M^1(\rho_3, \Lambda, \chi) = \frac{\xi(2z, \chi_6^2)\xi(z-\frac{1}{2}, \chi_6) \quad R(\rho_3, \Lambda, \chi)}{\xi(z+\frac{5}{2}, \chi_6)\xi(2z+1, \chi_6^2)\varepsilon(z+\frac{3}{2}, \chi_6)\varepsilon(z+\frac{1}{2}, \chi_6)\varepsilon(2z, \chi_6^2)\varepsilon(z-\frac{1}{2}, \chi_6)}$$

$$M^1\left(\sigma\left(\frac{\pi}{3}\right), \Lambda, \chi\right) = \frac{\xi(z+\frac{3}{2}, \chi_6)}{\xi(z+\frac{5}{2}, \chi_6)\varepsilon(z+\frac{3}{2}, \chi_6)} R\left(\sigma\left(\frac{\pi}{3}\right), \Lambda, \chi\right)$$

$$M^1\left(\sigma\left(\frac{2\pi}{3}\right), \Lambda, \chi\right)$$

$$= \frac{\xi(z+\frac{1}{2}, \chi_6)\xi(2z, \chi_6^2) \quad R\left(\sigma(\frac{2\pi}{3}), \Lambda, \chi\right)}{\xi(z+\frac{5}{2}, \chi_6)\xi(2z+1, \chi_6^2)\varepsilon(z+\frac{3}{2}\chi_6)\varepsilon(z+\frac{1}{2}, \chi_6)\varepsilon(z+\frac{1}{2}, \chi_6)\varepsilon(2z, \chi_6^2)}$$

$$M^1\left(\sigma(\pi), \Lambda, \chi\right) = \frac{\xi(2z, \chi_6^2)\xi(z-\frac{3}{2}, \chi_6)}{\xi(z+\frac{5}{2}, \chi_6)\xi(2z+1, \chi_6^2)}$$

$$\cdot \frac{R\left(\sigma(\pi), \Lambda, \chi\right)}{\varepsilon(z+\frac{3}{2}, \chi_6)\varepsilon(z+\frac{1}{2}, \chi_6)\varepsilon(2z, \chi_6^2)\varepsilon(z-\frac{1}{2}, \chi_6)\varepsilon(z-\frac{3}{2}, \chi_6)}$$

PROPOSITION 3.2.2. *If χ is not trivial, then* $\mathrm{Res}_{S_1} A(f, f'; \Lambda)$ *has a pole at $z = \frac{1}{2}$, i.e. $\Lambda = \beta_2$ when $\chi_6^2 = 1$, $\chi_6 \neq 1$ and the residue is given by*

$$\mathrm{Res}_{\beta_2} \mathrm{Res}_{S_1} A(f, f'; \Lambda) = c_1\big(f(\beta_2), R(\rho_2, \beta_2, \rho_2\chi)\big)$$

$$\left(R(\rho_5, \text{“}\beta_2\text{”}, \chi)f'(\beta_2) + c_2 R\left(\sigma\left(\frac{\pi}{3}\right), \beta_4, \rho_3\chi\right)f'(\beta_4) + c_3 R\left(\rho_6, \beta_4, \frac{2\pi}{3}\chi\right)f'(\beta_4)\right),$$

where c_1, c_2, c_3 are constants and $R(\rho_5, \text{“}\beta_2\text{”}, \chi) = \otimes_v R(\rho_5, \text{“}\beta_2\text{”}, \chi_v)$ and $R(\rho_5, \text{“}\beta_2\text{”}, \chi_v)$ is the value at β_2 of the restriction $R(\rho_5, \Lambda, \chi_v)|_{S_1}$. It is an isomorphism from $I(\beta_2, \chi)$ to $I(\beta_2, \rho_5\chi)$.

PROOF. The local intertwining operators are holomorphic at $z = \frac{1}{2}$ except $R\big(\sigma(\pi), \Lambda, \chi_v\big)$ when $\chi_v = 1$. However, Moeglin-Waldspurger [M-W1, Appendix III] showed that the restriction $R\big(\sigma(\pi), \Lambda, \chi_v\big)|_{S_1}$ is holomorphic at $z = \frac{1}{2}$: Let $\chi_v = 1$. By the cocycle relation, $R\big(\sigma(\pi), \Lambda, \chi_v\big) = R(\rho_5, \rho_2\Lambda, \chi_v)R(\rho_2, \Lambda, \chi_v)$. The pole is contained in $R(\rho_5, \rho_2\Lambda, \chi_v)$. However, the restriction $R(\rho_5, \rho_2\Lambda, \chi_v)|_{S_1}$ of $R(\rho_5, \rho_2\Lambda, \chi_v)$ onto S_1 is holomorphic at β_2. If χ_v is not trivial, $R(\rho_5, \rho_2\Lambda, \chi_v)$ is holomorphic at β_2.

Therefore, all the poles are contained in the normalizing factors. So

$$\begin{aligned}\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda) = {} & \frac{c(F)\xi(1,\chi_6)}{\xi(2)\xi(3,\chi_6)\varepsilon(1,\chi_6)\varepsilon(2,\chi_6)}\big(R(\rho_3,\beta_2,\chi)f(\beta_2),f'(\beta_4)\big)\\ & + \left(R\Big(\sigma\Big(\frac{2\pi}{3}\Big),\beta_2,\chi\Big)f(\beta_2),f'(\beta_4)\right)\\ & + \frac{c(F)\xi(2,\chi_6)}{\xi(2)\xi(3,\chi_6)\varepsilon(2,\chi_6)}\Big(R\big(\sigma(\pi),\beta_2,\chi\big)f(\beta_2),f'(\beta_2)\Big).\end{aligned}$$

In order to use the adjoint formula (3.2), we note that $\rho_3 = \rho_2\sigma(\frac{\pi}{3})$ and $\sigma(\frac{2\pi}{3})^{-1} = \sigma(\frac{4\pi}{3}) = \rho_2\rho_6$. We also note that $\sigma(\pi) = \rho_2\rho_5 = \rho_5\rho_2$ and so we have $R\big(\sigma(\pi),\Lambda,\chi_v\big) = R(\rho_2,\rho_5\Lambda,\rho_5\chi_v)R(\rho_5,\Lambda,\chi_v)$. By the same reasoning as above, if $\chi_v = 1$, $R(\rho_5,\Lambda,\chi_v)|_{S_1}$ is holomorphic at β_2 and we denote its value by $R(\rho_5,\text{“}\beta_2\text{”},\chi_v)$. If χ_v is not trivial, then $R(\rho_5,\Lambda,\chi_v)$ is holomorphic at β_2 and in this case, $R(\rho_5,\beta_2,\chi_v) = R(\rho_5,\text{“}\beta_2\text{”},\chi_v)$.

Proposition 3.2.3. *If χ is trivial, then* $\operatorname{Res}_{S_1} A(f,f';\Lambda)$ *has a pole at $z = \frac{1}{2}$, i.e. at β_2 and $z = \frac{3}{2}$, i.e., at β_3.*

(1) $\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda)$ *was calculated by Moeglin-Waldspurger.*

(2) $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda) = 0$.

Proof. (1) See [M-W1, Appendix IV].

(2) At $z = \frac{3}{2}$, $M^1(\rho_3,\Lambda,\chi)$ and $M^1\big(\sigma(\pi),\Lambda,\chi\big)$ have simple poles. So the residue is given by

$$\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda) = \frac{\xi(3)c(F)}{\xi(4)^2}\Big(\Big(R(\rho_3,\beta_3,\chi) - R\big(\sigma(\pi),\beta_3,\chi\big)\Big)f(\beta_3),f'(\beta_3)\Big).$$

But $R\big(\sigma(\pi),\beta_3,\chi\big) = R(\rho_3\rho_6,\beta_3,\chi) = R(\rho_3,\beta_3,\chi)R(\rho_6,\beta_3,\chi)$ and $R(\rho_6,\beta_3,\chi) = id$ by the following Lemma. So $\operatorname{Res}_{\beta_3}\operatorname{Res}_{S_1} A(f,f';\Lambda) = 0$.

Lemma 3.2.4. *Suppose a character χ_v satisfies $\rho_6\chi_v = \chi_v$. Then for any real number t, the normalized intertwining operator $R(\rho_6,t\beta_3,\chi_v)$ is a self-intertwining operator of $I(t\beta_3,\chi_v)$. It acts like the identity.*

Proof. Since $\rho_6\beta_3 = \beta_3$, it is a self-intertwining operator of $I(t\beta_3,\chi_v)$. Since $\langle\beta_3,\beta_6^\vee\rangle = 0$, $R(\rho_6,t\beta_3,\chi_v)$ is actually an intertwining operator for the Levi subgroup $M_2 \simeq \mathrm{GL}_2$, where $P_2 = M_2N_2$ is the maximal parabolic subgroup attached to β_6. Under the identification $M_2 \simeq \mathrm{GL}_2$, $R(\rho_6,t\beta_3,\chi_v)$ is an intertwining operator of $\mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\mu|\ |^t \times \mu|\ |^t$. Since $\mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\mu|\ |^t \times \mu|\ |^t$ is irreducible, $R(\rho_6,t\beta_3,\chi_v)$ acts like a scalar. But it acts like the identity on the K-fixed vector. Therefore it is the identity.

3.3. *Calculation of* $\operatorname{Res}_{S_2} A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_2 if $\chi_2 = 1$, *i.e.*, $\chi_1^3\chi_6 = 1$.

On S_2,

$$\langle\Lambda,\beta_1^\vee\rangle = -z + \frac{1}{2}, \langle\Lambda,\beta_3^\vee\rangle = z + \frac{1}{2}$$
$$\langle\Lambda,\beta_4^\vee\rangle = 3z + \frac{1}{2}, \langle\Lambda,\beta_5^\vee\rangle = 2z, \langle\Lambda,\beta_6^\vee\rangle = 3z - \frac{1}{2}.$$

LEMMA 3.3.1.

$$M^2(\rho_2,\Lambda,\chi)=\frac{\xi(-z+\frac{1}{2},\chi_1)\xi(z+\frac{1}{2},\chi_1^{-1})\quad R(\rho_2,\Lambda,\chi)}{\xi(-z+\frac{3}{2},\chi_1)\xi(z+\frac{3}{2},\chi_1^{-1})\varepsilon(-z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{1}{2},\chi_1^{-1})}$$

$$M^2(\rho_3,\Lambda,\chi)=\frac{\xi(-z+\frac{1}{2},\chi_1)\xi(z+\frac{1}{2},\chi_1^{-1})\xi(3z+\frac{1}{2},\chi_1^{-3})\xi(2z,\chi_1^{-2})}{\xi(-z+\frac{3}{2},\chi_1)\xi(z+\frac{3}{2},\chi_1^{-1})\xi(3z+\frac{3}{2},\chi_1^{-3})\xi(2z+1,\chi_1^{-2})}$$
$$\cdot\frac{R(\rho_3,\Lambda,\chi)}{\varepsilon(-z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{1}{2},\chi_1^{-1})\varepsilon(3z+\frac{1}{2},\chi_1^{-3})\varepsilon(2z,\chi_1^{-2})}$$

$$M^2(\rho_4,\Lambda,\chi)=\frac{\xi(z+\frac{1}{2},\chi_1^{-1})\xi(2z,\chi_1^{-2})\xi(3z-\frac{1}{2},\chi_1^{-3})}{\xi(z+\frac{3}{2},\chi_1^{-1})\xi(2z+1,\chi_1^{-2})\xi(3z+\frac{3}{2},\chi_1^{-3})}$$
$$\cdot\frac{R(\rho_4,\Lambda,\chi)}{\varepsilon(z+\frac{1}{2},\chi_1^{-1})\varepsilon(3z+\frac{1}{2},\chi_1^{-3})\varepsilon(2z,\chi_1^{-2})\varepsilon(3z-\frac{1}{2},\chi_1^{-3})}$$

$$M^2\left(\sigma\left(\frac{\pi}{3}\right),\Lambda,\chi\right)=\frac{\xi(-z+\frac{1}{2},\chi_1)}{\xi(-z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{1}{2},\chi_1)}R\left(\sigma\left(\frac{\pi}{3}\right),\Lambda,\chi\right)$$

$$M^2\left(\sigma\left(\frac{2\pi}{3}\right),\Lambda,\chi\right)=\frac{\xi(-z+\frac{1}{2},\chi_1)\xi(z+\frac{1}{2},\chi_1^{-1})\xi(3z+\frac{1}{2},\chi_1^{-3})}{\xi(-z+\frac{3}{2},\chi_1)\xi(z+\frac{3}{2},\chi_1^{-1})\xi(3z+\frac{3}{2},\chi_1^{-3})}$$
$$\cdot\frac{R\left(\sigma(\frac{2\pi}{3}),\Lambda,\chi\right)}{\varepsilon(-z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{1}{2},\chi_1^{-1})\varepsilon(3z+\frac{1}{2},\chi_1^{-3})}$$

$$M^2\left(\sigma(\pi),\Lambda,\chi\right)=\frac{\xi(-z+\frac{1}{2},\chi_1)\xi(z+\frac{1}{2},\chi_1^{-1})\xi(2z,\chi_1^{-2})\xi(3z-\frac{1}{2},\chi_1^{-3})}{\xi(-z+\frac{3}{2},\chi_1)\xi(z+\frac{3}{2},\chi_1^{-1})\xi(2z+1,\chi_1^{-2})\xi(3z+\frac{3}{2},\chi_1^{-3})}$$
$$\cdot\frac{R\left(\sigma(\pi),\Lambda,\chi\right)}{\varepsilon(-z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{1}{2},\chi_1^{-1})\varepsilon(2z,\chi_1^{-2})\varepsilon(3z+\frac{1}{2},\chi_1^{-3})\varepsilon(3z-\frac{1}{2},\chi_1^{-3})}$$

PROPOSITION 3.3.2. $\mathrm{Res}_{S_2} A(f,f';\Lambda)$ *has a simple pole at* $z=\frac{1}{6}$, *i.e.* $\Lambda=\frac{1}{3}\beta_3$ *when* $\chi_1^3=1$, *i.e.,* $\chi_6=1$ *and* $\mathrm{Res}_{\frac{1}{3}\beta_3}\mathrm{Res}_{S_2} A(f,f';\Lambda)=0$.

PROOF. All the local intertwining operators are holomorphic at $z=\frac{1}{6}$. So all the poles are contained in the normalizing factors. Therefore,

$$\mathrm{Res}_{\frac{1}{3}\beta_3}\mathrm{Res}_{S_2} A(f,f';\Lambda)$$
$$=(*)\left(\left(R\left(\rho_3,\frac{1}{3}\beta_3,\chi\right)-R\left(\sigma(\pi),\frac{1}{3}\beta_3,\chi\right)\right)f\left(\frac{1}{3}\beta_3\right),f'\left(\frac{1}{3}\beta_3\right)\right)$$
$$+(**)\left(\left(-R\left(\rho_4,\frac{1}{3}\beta_3,\chi\right)+R\left(\sigma\left(\frac{2\pi}{3}\right),\frac{1}{3}\beta_3,\chi\right)\right)f\left(\frac{1}{3}\beta_3\right),f'\left(\frac{1}{3}\beta_3\right)\right),$$

with $(*)$ and $(**)$ being constants depending on χ_1. Here $R\left(\sigma(\pi),\frac{1}{3}\beta_3,\chi\right)=R(\rho_3,\frac{1}{3}\beta_3,\chi)R(\rho_6,\frac{1}{3}\beta_3,\chi)$ and $R(\rho_4,\frac{1}{3}\beta_3,\chi)=R\left(\sigma(\frac{2\pi}{3}),\frac{1}{3}\beta_3,\chi\right)R(\rho_6,\frac{1}{3}\beta_3,\chi)$. Since $\rho_6\chi=\chi$, by Lemma 3.2.4, $R(\rho_6,\frac{1}{3}\beta_3,\chi)=id$. Therefore, $\mathrm{Res}_{\frac{1}{3}\beta_3}\mathrm{Res}_{S_2} A(f,f';\Lambda)=0$.

3.4. *Calculation of* $\operatorname{Res}_{S_5} A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_5 if $\chi_5 = 1$, *i.e.* $\chi_1\chi_6 = 1$. On S_5,

$$\langle\Lambda,\beta_1^\vee\rangle = z-\frac{1}{2}, \langle\Lambda,\beta_2^\vee\rangle = 2z, \langle\Lambda,\beta_3^\vee\rangle = z+\frac{1}{2}$$
$$\langle\Lambda,\beta_4^\vee\rangle = z+\frac{3}{2}, \langle\Lambda,\beta_6^\vee\rangle = -z+\frac{3}{2}.$$

LEMMA 3.4.1.

$$M^5(\rho_3,\Lambda,\chi) = \frac{\xi(z-\frac{1}{2},\chi_1)\xi(2z,\chi_1^2) \quad R(\rho_3,\Lambda,\chi)}{\xi(z+\frac{5}{2},\chi_1)\xi(2z+1,\chi_1^2)\varepsilon(z-\frac{1}{2},\chi_1)\varepsilon(2z,\chi_1^2)\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)}$$

$$M^5(\rho_4,\Lambda,\chi) = \frac{\xi(2z,\chi_1^2)\xi(z+\frac{1}{2},\chi_1)\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(2z+1,\chi_1^2)\xi(z+\frac{5}{2},\chi_1)\xi(-z+\frac{5}{2},\chi_1^{-1})}$$
$$\cdot\frac{R(\rho_4,\Lambda,\chi)}{\varepsilon(2z,\chi_1^2)\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5(\rho_5,\Lambda,\chi) = \frac{\xi(z+\frac{3}{2},\chi_1)\xi(-z+\frac{3}{2},\chi_1^{-1}) \quad R(\rho_5,\Lambda,\chi)}{\xi(z+\frac{5}{2},\chi_1)\xi(-z+\frac{5}{2},\chi_1^{-1})\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5\big(\sigma(\pi),\Lambda,\chi\big) = \frac{\xi(z-\frac{1}{2},\chi_1)\xi(2z,\chi_1^2)\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(z+\frac{5}{2},\chi_1)\xi(2z+1,\chi_1^2)\xi(-z+\frac{5}{2},\chi_1^{-1})}$$
$$\cdot\frac{R\big(\sigma(\pi),\Lambda,\chi\big)}{\varepsilon(z-\frac{1}{2},\chi_1)\varepsilon(2z,\chi_1^2)\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5\Big(\sigma\Big(\frac{4\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(z+\frac{1}{2},\chi_1)\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(z+\frac{5}{2},\chi_1)\xi(-z+\frac{5}{2},\chi_1^{-1})}$$
$$\cdot\frac{R\big(\sigma(\frac{4\pi}{3}),\Lambda,\chi\big)}{\varepsilon(z+\frac{1}{2},\chi_1)\varepsilon(z+\frac{3}{2},\chi_1)\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}$$

$$M^5\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(-z+\frac{3}{2},\chi_1^{-1})}{\xi(-z+\frac{5}{2},\chi_1^{-1})\varepsilon(-z+\frac{3}{2},\chi_1^{-1})}R\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big)$$

PROPOSITION 3.4.2. *If χ is not trivial, $\operatorname{Res}_{S_5} A(f,f';\Lambda)$ has a simple pole at $\Lambda = \beta_4$, i.e., $z = \frac{1}{2}$, when $\chi_1^2 = 1$, $\chi_1 \neq 1$ and its residue is given by*

$$\operatorname{Res}_{\beta_4}\operatorname{Res}_{S_5} A(f,f';\Lambda) = c_1\big(R(\rho_3,\beta_4,\chi)f(\beta_4),f'(\beta_2)\big)$$
$$+c_2\Big(\Big(R(\rho_4,\beta_4,\chi)+R\big(\sigma(\pi),\beta_4,\chi\big)\Big)f(\beta_4),f'(\beta_4)\Big),$$

where $c_1 = \frac{c(F)\xi(0,\chi_1)}{\xi(2)\xi(3,\chi_1)\varepsilon(2,\chi_1)}$ *and* $c_2 = \frac{c(F)\xi(1,\chi_1)^2}{\xi(2)\xi(3,\chi_1)\xi(2,\chi_1)\varepsilon(1,\chi_1)^2}$.

PROOF. All the local intertwining operators are holomorphic at $z = \frac{1}{2}$. Therefore all the poles are contained in the normalizing factors. Our assertion follows from the straightforward computation.

PROPOSITION 3.4.3. *If χ is trivial,* $\mathrm{Res}_{S_5} A(f,f';\Lambda)$ *has a triple pole at* $z = \frac{1}{2}$. *Moeglin-Waldspurger calculated the residue.*

3.5. *Calculation of* $\mathrm{Res}_{S_6} A(f,f';\Lambda)$. $M(w,\Lambda,\chi)$ has a pole at S_6 if $\chi_6 = 1$. On S_6,

$$\langle\Lambda,\beta_1^\vee\rangle = z - \frac{1}{2}, \quad \langle\Lambda,\beta_2^\vee\rangle = 3z - \frac{1}{2}, \quad \langle\Lambda,\beta_3^\vee\rangle = 2z,$$
$$\langle\Lambda,\beta_4^\vee\rangle = 3z + \frac{1}{2}, \quad \langle\Lambda,\beta_5^\vee\rangle = z + \frac{1}{2}.$$

LEMMA 3.5.1.

$$M^6(\rho_4,\Lambda,\chi) = \frac{\xi(3z-\frac{1}{2},\chi_1^3)\xi(2z,\chi_1^2)\xi(z+\frac{1}{2},\chi_1)}{\xi(3z+\frac{3}{2},\chi_1^3)\xi(2z+1,\chi_1^2)\xi(z+\frac{3}{2},\chi_1)}$$
$$\cdot \frac{R(\rho_4,\Lambda,\chi)}{\varepsilon(3z-\frac{1}{2},\chi_1^3)\varepsilon(2z,\chi_1^2)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6(\rho_5,\Lambda,\chi) = \frac{\xi(3z+\frac{1}{2},\chi_1^3)\xi(z+\frac{1}{2},\chi_1) \quad R(\rho_5,\Lambda,\chi)}{\xi(3z+\frac{3}{2},\chi_1^3)\xi(z+\frac{3}{2},\chi_1)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6(\rho_6,\Lambda,\chi) = R(\rho_6,\Lambda,\chi)$$

$$M^6\big(\sigma(\pi),\Lambda,\chi\big) = \frac{\xi(z-\frac{1}{2},\chi_1)\xi(3z-\frac{1}{2},\chi_1^3)\xi(2z,\chi_1^2)}{\xi(z+\frac{3}{2},\chi_1)\xi(3z+\frac{3}{2},\chi_1^3)\xi(2z+1,\chi_1^2)}$$
$$\cdot \frac{R\big(\sigma(\pi),\Lambda,\chi\big)}{\varepsilon(z-\frac{1}{2},\chi_1)\varepsilon(3z-\frac{1}{2},\chi_1^3)\varepsilon(2z,\chi_1^2)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6\Big(\sigma\Big(\frac{4\pi}{3}\Big),\Lambda,\chi\Big)$$
$$= \frac{\xi(2z,\chi_1^2)\xi(3z+\frac{1}{2},\chi_1^3)\xi(z+\frac{1}{2},\chi_1) \quad R\big(\sigma(\frac{4\pi}{3}),\Lambda,\chi\big)}{\xi(2z+1,\chi_1^2)\xi(3z+\frac{3}{2},\chi_1^3)\xi(z+\frac{3}{2},\chi_1)\varepsilon(2z,\chi_1^2)\varepsilon(3z+\frac{1}{2},\chi_1^3)\varepsilon(z+\frac{1}{2},\chi_1)}$$

$$M^6\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big) = \frac{\xi(z+\frac{1}{2},\chi_1)}{\xi(z+\frac{3}{2},\chi_1)\varepsilon(z+\frac{1}{2},\chi_1)} R\Big(\sigma\Big(\frac{5\pi}{3}\Big),\Lambda,\chi\Big).$$

All the local intertwining operators are holomorphic at $z = \frac{1}{6}, z = \frac{1}{2}, z = \frac{3}{2}$ and so all the poles are contained in the normalizing factors.

PROPOSITION 3.5.2. *If* χ *is not trivial,* $\mathrm{Res}_{S_6} A(f,f';\Lambda)$ *has a simple pole at* $\Lambda = \frac{1}{3}\beta_5$, *i.e.,* $z = \frac{1}{6}$ *if* $\chi_1^3 = 1$. $\mathrm{Res}_{\frac{1}{3}\beta_5} \mathrm{Res}_{S_6} A(f,f';\Lambda) = 0$.

PROOF.

$$\mathrm{Res}_{\frac{1}{3}\beta_5} \mathrm{Res}_{S_6} A(f,f';\Lambda)$$
$$= c_1\bigg(\Big(-R\Big(\rho_4,\frac{1}{3}\beta_5,\chi\Big) + R\Big(\sigma\Big(\frac{4\pi}{3}\Big),\frac{1}{3}\beta_5,\chi\Big)\Big)f\Big(\frac{1}{3}\beta_5\Big), f'\Big(\frac{1}{3}\beta_3\Big)\bigg)$$
$$+ c_2\bigg(\Big(R\Big(\rho_5,\frac{1}{3}\beta_5,\chi\Big) - R\Big(\sigma(\pi),\frac{1}{3}\beta_5,\chi\Big)\Big)f\Big(\frac{1}{3}\beta_5\Big), f'\Big(\frac{1}{3}\beta_5\Big)\bigg)$$

where c_1, c_2 depend on χ_1. But

$$R\Big(\rho_4, \frac{1}{3}\beta_5, \chi\Big) = R\Big(\rho_6, -\frac{1}{3}\beta_3, \chi\Big) R\Big(\sigma\Big(\frac{4\pi}{3}\Big), \frac{1}{3}\beta_5, \chi\Big)$$
$$R\Big(\sigma(\pi), \frac{1}{3}\beta_5, \chi\Big) = R\Big(\rho_5, \frac{1}{3}\beta_5, \chi\Big) R\Big(\rho_2, \frac{1}{3}\beta_5, \chi\Big).$$

Here $\sigma(\frac{4\pi}{3})\chi = \chi$ and $\rho_6\rho_1\chi = \rho_1\chi$ and

$$R\Big(\rho_2, \frac{1}{3}\beta_5, \chi\Big) = R\Big(\rho_1, \frac{1}{3}\beta_3, \rho_1\chi\Big) R\Big(\rho_6, \frac{1}{3}\beta_3, \rho_1\chi\Big) R\Big(\rho_1, \frac{1}{3}\beta_5, \chi\Big).$$

By Lemma 3.2.4, $R(\rho_6, -\frac{1}{3}\beta_3, \chi)$ and $R(\rho_6, \frac{1}{3}\beta_3, \rho_1\chi)$ are the identity. By the cocycle relation of the normalized intertwining operators, $R(\rho_2, \frac{1}{3}\beta_5, \chi)$ is also the identity. Therefore, the residue is zero.

PROPOSITION 3.5.3. *$A(f,f';\Lambda)$ has a simple pole at $\Lambda = \beta_4$, i.e., $z = \frac{1}{2}$ if $\chi_1^3 = 1$, $\chi_1^2 \neq 1$ or $\chi_1^2 = 1$, $\chi_1 \neq 1$.*

(1) $\chi_1^3 = 1$, $\chi_1^2 \neq 1$. The residue is given by

$$\mathrm{Res}_{\beta_4} \mathrm{Res}_{S_6} A(f,f';\Lambda) = c\Big(f(\beta_4), R(\rho_4, \beta_4, \rho_4\chi)(f'(\beta_4) + R(\rho_1, \beta_4, \chi)f'(\beta_4)\Big),$$

where $c = \frac{c(F)\xi(1,\chi_1^2)\xi(1,\chi_1)}{\xi(2)\xi(2,\chi_1^2)\xi(2,\chi_1)\epsilon(1,\chi_1^2)\epsilon(1,\chi_1)}$.

(2) $\chi_1^2 = 1$, $\chi_1 \neq 1$. The residue is given by

$$\mathrm{Res}_{\beta_4} \mathrm{Res}_{S_6} A(f,f';\Lambda) = c_1\Big(\Big(R(\rho_4, \beta_4, \chi) + R\big(\sigma(\pi), \beta_4, \chi\big)\Big)f(\beta_4), f'(\beta_4)\Big)$$
$$+ c_2\left(R\Big(\sigma\Big(\frac{4\pi}{3}\Big), \beta_4, \chi\Big)f(\beta_4), f'(\beta_2)\right),$$

where $c_1 = \frac{\xi(1,\chi_1)\xi(1,\chi_1)\quad c(F)}{\xi(2,\chi_1)\xi(2,\chi_1)\epsilon(1,\chi_1)\epsilon(2,\chi_1)\epsilon(1,\chi_1)\xi(2)}$ and $c_2 = \frac{c(F)\xi(1,\chi_1)}{\xi(2)\xi(3,\chi_1)\epsilon(2,\chi_1)\epsilon(1,\chi_1)}$.

PROOF. It follows from Lemma 3.5.1 and straightforward computation.

PROPOSITION 3.5.4. *If χ is trivial,* $\mathrm{Res}_{S_6} A(f,f';\Lambda)$ *has a triple pole at $\Lambda = \beta_4$. Moeglin-Waldspurger calculated the residue.*

PROPOSITION 3.5.5. *If χ is trivial,* $\mathrm{Res}_{S_6} A(f,f';\Lambda)$ *has a simple pole at $\Lambda = \rho_B$, i.e., $z = \frac{3}{2}$. The residue is given by*

$$\mathrm{Res}_{\rho_B} \mathrm{Res}_{S_6} A(f,f';\Lambda) = \frac{c(F)\xi(\frac{5}{2})}{\xi(\frac{9}{2})\xi(4)}\Big(R\big(\sigma(\pi), \rho_B, \chi\big)f(\rho_B), f'(\rho_B)\Big).$$

This gives the constant.

3.6. *Conclusion.* Let $J(\chi)$ be the subspace of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$, which is K-finite and K_∞-invariant.

3.6.1. χ is trivial. We can see from the above calculation (Propositions 3.2.3, 3.3.2, 3.4.3, 3.5.4, 3.5.5) that Moeglin and Waldspurger obtained all the residual spectrum attached to the trivial character of the torus. We summarize their results; $J(1)$ is isomorphic to the sum of the trivial representation and the image $(1+\frac{1}{2}E)R(\rho_2,\beta_2)I(\beta_2)_f$, where $I(\beta_2)_f$ is the K_∞-invariant subspace of $I(\beta_2)$ and $E = \otimes_v E_v$. Here E_v is defined as follows: Let $R(\rho_5,\rho_2\Lambda)|_{S_1}$ be the restriction of $R(\rho_5,\rho_2\Lambda)$ to S_1. It is holomorphic at β_2. Let E_v be the value of $R(\rho_5,\rho_2\Lambda)|_{S_1}$ at β_2.

Then $R_v(\rho_2,\beta_2)I_v(\beta_2) = \pi_{1v} \oplus \pi_{2v}$, where π_{1v} is spherical and

$$E_v(f_v) = \begin{cases} f_v, & \text{if } f_v \in \pi_{1v} \\ -2f_v, & \text{if } f_v \in \pi_{2v}. \end{cases} \tag{3.3}$$

Let S be a finite set of finite places and $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$. Then

$$J(1) = \pi_0 \oplus \bigoplus_{S,\mathrm{card}(S)\neq 1} \pi^S,$$

where π_0 is the trivial representation.

3.6.2. χ is non-trivial. From Propositions 3.2.2, 3.4.2 and 3.5.3, only the following characters contribute to the residual spectrum:

(1) $\chi\colon \chi_1 = 1, \chi_6^2 = 1, \chi_6 \neq 1$
(2) $\chi' = \rho_6\chi\colon {\chi_1'}^2 = 1, \chi_1' \neq 1, \chi_6' = \chi_1'$
(3) $\chi'' = \rho_1\rho_6\chi\colon {\chi_1''}^2 = 1, \chi_1'' \neq 1, \chi_6'' = 1$
(4) $\tilde{\chi}\colon \tilde{\chi}^3 = 1, \tilde{\chi}_1 \neq 1, \tilde{\chi}_6 = 1$

Under the identification $M_1 \simeq \mathrm{GL}_2$, where M_1 is the Levi subgroup of P_1, the above characters are given by $\chi = \chi(\mu,\nu)$, where μ,ν are grössencharacters of F:

(1) $\chi = \chi(\mu,\nu), \mu = \nu, \mu^2 = 1, \mu \neq 1$
(2) $\rho_6\chi = \chi(\mu,\nu), \mu = 1, \nu^2 = 1, \nu \neq 1$
(3) $\rho_1\rho_6\chi = \chi(\mu,\nu), \mu^2 = 1, \mu \neq 1, \nu = 1$
(4) $\tilde{\chi} = \chi(\mu,\nu), \mu^3 = 1, \mu \neq 1, \nu = \mu^2$

CASE 1. $\chi\colon \chi_1 = 1, \chi_6^2 = 1, \chi_6 \neq 1$

From Proposition 3.2.2,

$$\operatorname{Res}_{\beta_2}\operatorname{Res}_{S_1} A(f,f';\Lambda) = c_1\Big(f(\beta_2), R(\rho_2,\beta_2,\rho_2\chi)$$
$$\left(R(\rho_5,\text{“}\beta_2\text{''},\chi)f'(\beta_2) + c_2R\Big(\sigma\Big(\frac{\pi}{3}\Big),\beta_4,\rho_3\chi\Big)f'(\beta_4) + c_3R\Big(\rho_6,\beta_4,\frac{2\pi}{3}\chi\Big)f'(\beta_4)\right),$$

where c_1, c_2, c_3 are constants and $R(\rho_5,\text{“}\beta_2\text{”},\chi) = \otimes_v R(\rho_5,\text{“}\beta_2\text{”},\chi_v)$ and $R(\rho_5,\text{“}\beta_2\text{”},\chi_v)$ is the value at β_2 of the restriction $R(\rho_5,\Lambda,\chi_v)|_{S_1}$. It is an isomorphism from $I(\beta_2,\chi)$ to $I(\beta_2,\rho_5\chi)$.

Here we recall the inner product formula (3.1) and our short-hand notation. Note that $\sigma(\pi) \in W(\chi,\chi)$, $\rho_3 \in W(\chi,\rho_6\chi)$ and $\sigma(\frac{2\pi}{3}) \in W(\chi,\rho_1\rho_6\chi)$. Therefore, the above f''s are all in different spaces. Since $R\big(\sigma(\frac{\pi}{3}),\beta_4,\rho_3\chi\big)$ and $R(\rho_6,\beta_4,\frac{2\pi}{3}\chi)$ are intertwining operators and $R(\rho_5, ``\beta_2", \chi)$ is an isomorphism, $J(\chi)$ is isomorphic to the image

$$R(\rho_2,\beta_2,\rho_2\chi)I(\beta_2,\rho_2\chi)_f = \bigotimes_{v<\infty} R_v(\rho_2,\beta_2,\rho_2\chi_v)I_v(\beta_2,\rho_2\chi_v).$$

We already know from Moeglin-Waldspurger that if χ_v is trivial,

$$R_v(\rho_2,\beta_2,\rho_2\chi_v)I_v(\beta_2,\rho_2\chi_v) = \pi_{1v} \oplus \pi_{2v}.$$

Suppose χ_v is not trivial. Then $\chi_v \otimes \exp\big(\beta_2, H_B()\big)$ is a regular character of T. So we can apply Rodier's result as follows: By the cocycle relation,

$$R_v(\rho_2,\beta_2,\rho_2\chi_v) = R_v(\rho_5,-\beta_2,\rho_5\chi_v)R_v\big(\sigma(\pi),\beta_2,\rho_2\chi_v\big).$$

From Rodier [R, Cor 3 in p417], $R_v\big(\sigma(\pi),\beta_2,\rho_2\chi_v\big)I_v(\beta_2,\rho_2\chi_v)$ and the unique irreducible subrepresentation of $I_v(-\beta_2,\rho_2\chi_v)$ have the same Jacquet module. Therefore, $R_v\big(\sigma(\pi),\beta_2,\rho_2\chi_v\big)I_v(\beta_2,\rho_2\chi_v)$ is the unique irreducible subrepresentation of $I_v(-\beta_2,\rho_2\chi_v)$. Since $R_v(\rho_5,-\beta_2,\rho_5\chi_v)$ is an isomorphism, $R_v(\rho_2,\beta_2,\rho_2\chi_v)I_v(\beta_2,\rho_2\chi_v)$ is the unique irreducible subrepresentation of $I(-\beta_2,\chi_v)$.

Let $J_v = \{\pi_{1v},\pi_{2v}\}$. If χ_v is not trivial, we take $\pi_{2v} = 0$. Let S be a finite set of finite places and $\pi^S = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$. Then

$$J(\chi) = \bigoplus_S \pi^S.$$

There is no condition on S.

CASE 2. $\chi' = \rho_6\chi : {\chi'_1}^2 = 1, \chi'_1 \neq 1, \chi'_6 = \chi'_1$
From Proposition 3.4.2 and the adjoint formula (3.2),

$$\begin{aligned}
&\mathrm{Res}_{\beta_4}\mathrm{Res}_{S_5} A(f,f';\Lambda)\\
&\quad = c_1\big(f(\beta_4), R(\rho_3,\beta_2,\rho_3\chi')f'(\beta_2)\big)\\
&\qquad + c_2\big(f(\beta_4), R(\rho_4,\beta_4,\rho_4\chi')\big)\big(f'(\beta_4) + R(\rho_1,\beta_4,\chi')f'(\beta_4)\big).
\end{aligned}$$

Recall the inner product formula (3.1). We note that $\rho_4 \in W(\rho_6\chi,\rho_1\rho_6\chi)$ since $\rho_4\rho_6\chi = \rho_1\rho_6\chi$ and $\sigma(\pi) \in W(\rho_6\chi,\rho_6\chi)$, $\rho_3 \in W(\rho_6\chi,\chi)$. Therefore, the above f''s are all in different spaces. Here

$$R(\rho_4,\beta_4,\rho_4\chi')I(\beta_4,\rho_4\chi') = \bigotimes_v R_v(\rho_4,\beta_4,\rho_4\chi'_v)I_v(\beta_4,\rho_v\chi'_v).$$

Under the identification $M_1 \simeq \mathrm{GL}_2$, $\rho_6\chi = \chi(\mu,\nu)$, $\mu = 1$, $\nu^2 = 1$, $\nu \neq 1$. By inducing in stages, $I(\beta_4,\rho_4\chi'_v) = \mathrm{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes \mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\rho_4\chi'_v$, where B_0 is a Borel subgroup of GL_2. $\pi_v = \mathrm{Ind}_{B_0}^{\mathrm{GL}_2}\rho_4\chi'_v$ is irreducible. Therefore, $R_v(\rho_4,\beta_4,\rho_4\chi'_v)I_v(\beta_4,\rho_v\chi'_v)$ is the Langlands' quotient of $\mathrm{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes \pi_v$. In particular, it is irreducible.

We have $\rho_3 = \rho_1\rho_4\rho_6$ and $\rho_4\chi' = \chi$. So

$$R(\rho_3,\beta_2,\chi_v) = R(\rho_1,-\beta_4,\rho_4\rho_6\chi_v)R(\rho_4,\beta_4,\rho_6\chi_v)R(\rho_6,\beta_2,\chi_v).$$

If χ_v is not trivial, then $R(\rho_1, -\beta_4, \rho_4\rho_6\chi_v)$ and $R(\rho_6, \beta_2, \chi_v)$ are isomorphisms. Therefore, the image of $R(\rho_3, \beta_2, \chi_v)$ is irreducible.

If χ_v is trivial, $R(\rho_6, \beta_2, \chi_v)$ is not an isomorphism and we proceed as follows: Since $\rho_3 = \rho_1\sigma(\frac{2\pi}{3})$, $R(\rho_3, \beta_2, \chi_v) = R\big(\rho_1, -\beta_4, \sigma(\frac{2\pi}{3}), \chi_v\big)R\big(\sigma(\frac{2\pi}{3}), \beta_2, \chi_v\big)$. As in Lemma 3.2.4, we can show that $R\big(\rho_1, -\beta_4, \sigma(\frac{2\pi}{3})\chi_v\big)$ is the identity. Also from [M-W1, equation (15)], $R_v(\rho_6, -\beta_2)\big(R_v(\rho_2, \beta_2) - E_vR_v(\rho_2, \beta_2)\big) = 0$. From (3.3), $E_vf = -2f$ for $f \in \pi_{2v}$ and so $R_v(\rho_6, -\beta_2)f = 0$ for $f \in \pi_{2v}$. Therefore $R_v(\rho_6, -\beta_2)R_v(\beta_2, \beta_2)I(\beta_2)$ is irreducible. Since $\rho_6\rho_2 = \sigma(\frac{2\pi}{3})$, the image of $R_v\big(\sigma(\frac{2\pi}{3}), \beta_2\big)$ is irreducible. It is isomorphic to the image $R_v(\rho_4, \beta_4)I(\beta_4)$.

Therefore, $J(\chi')$ is isomorphic to the image $R(\rho_4, \beta_4, \rho_4\chi')I(\beta_4, \rho_4\chi')$. Since $\rho_6\beta_4 = \beta_2$, $J(\chi')$ is isomorphic to none other than $\otimes_v \pi_{1v}$ in Case 1.

CASE 3. $\chi'' = \rho_1\rho_6\chi : \chi_1''^2 = 1, \chi_1'' \neq 1, \chi_6'' = 1$
From Proposition 3.5.3 and the adjoint formula (3.2),

$$\operatorname{Res}_{\beta_4} \operatorname{Res}_{S_6} A(f, f'; \Lambda) = c_1(f(\beta_4), R(\rho_4, \beta_4, \rho_4\chi'')\big(f'(\beta_4) + R(\rho_1, \beta_4, \chi'')f'(\beta_4)\big)$$
$$+ c_2\left(f(\beta_4), R\left(\sigma\left(\frac{2\pi}{3}\right), \beta_2, \sigma\left(\frac{4\pi}{3}\right)\chi''\right)f'(\beta_2)\right),$$

We proceed in the same way as Case 2. $J(\chi'')$ is isomorphic to the image $R(\rho_4, \beta_4, \rho_4\chi'')I(\beta_4, \rho_4\chi'')$. It is irreducible and it is isomorphic to the one in Case 2.

CASE 4. $\tilde{\chi} : \tilde{\chi}_1^3 = 1, \tilde{\chi}_1^2 \neq 1, \tilde{\chi}_6 = 1$
From Proposition 3.5.3,

$$\begin{aligned} \text{residue} &= \operatorname{Res}_{\beta_4} \operatorname{Res}_{S_6} A(f, f'; \Lambda) \\ &= c\big(f(\beta_4), R(\rho_4, \beta_4, \rho_4\tilde{\chi})\big(f'(\beta_4) + R(\rho_1, \beta_4, \tilde{\chi})f'(\beta_4)\big). \end{aligned}$$

So $J(\tilde{\chi})$ is isomorphic to the image $R(\rho_4, \beta_4, \rho_4\tilde{\chi})I(\beta_4, \rho_4\tilde{\chi})$. By inducing in stages, $I(\beta_4, \rho_4\tilde{\chi}_v) = \operatorname{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes \operatorname{Ind}_{B_0}^{\mathrm{GL}_2} \rho_4\tilde{\chi}_v$, where B_0 is a Borel subgroup of GL_2. $\pi_v = \operatorname{Ind}_{B_0}^{\mathrm{GL}_2} \rho_4\tilde{\chi}_v$ is irreducible. Therefore, $R_v(\rho_4, \beta_4, \rho_4\tilde{\chi}_v)I_v(\beta_4, \rho_v\tilde{\chi}_v)$ is the Langlands' quotient of $\operatorname{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes \pi_v$. So $J(\tilde{\chi})$ is irreducible.

We have proved

THEOREM 3.6.1. *Let $J(\chi)$ be the subspace of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{(T,\chi)}$, which is K-finite, K_∞-invariant. Then the K-finite, K_∞-invariant subspace of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$ is the direct sum of the following space:*

(1) $J(1) = \pi_0 \oplus \bigoplus_{S, \mathrm{card}(S) \neq 1} \pi^S$, where $\pi^S = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$, π_{1v} is spherical.

(2) $J(\chi) = \bigoplus_S \pi^S$, where $\chi_1 = 1$, $\chi_6^2 = 1$, $\chi_6 \neq 1$. $\pi^S = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$. If χ_{6v} is not trivial, we set $\pi_{2v} = 0$.

(3) $J(\tilde{\chi}) =$ the Langlands' quotient of $\operatorname{Ind}_{P_1}^G \exp\big(\beta_4, H_{P_1}()\big) \otimes (\operatorname{Ind}_{B_0}^{\mathrm{GL}_2} \tilde{\chi})$, where $\tilde{\chi}^3 = 1$, $\tilde{\chi}_1 \neq 1$, $\tilde{\chi}_6 = 1$.

3.7. *Arthur Parameters.* In this section, we give the Arthur parameters for the residual spectrum of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_T$. We say that a unipotent element u is distinguished if all maximal tori of Cent(u, G) are contained in the center of G°, the connected component of the identity. This is equivalent to the fact that the unipotent orbit O of u does not meet any proper Levi subgroup of G (Spaltenstein [Sp, p67]). (*i.e.*, if L is a Levi subgroup of a parabolic subgroup of G and $u \in L$ for a $u \in O$, then $L^\circ = G^\circ$.)

JACOBSON-MOROZOV THEOREM. *Suppose u is a unipotent element in a semi-simple algebraic group G. Then there exists a homomorphism $\phi\colon \mathrm{SL}_2 \mapsto G$ such that $\phi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = u$.*

LEMMA ([B-V, PROP. 2.4]). *Let u be a unipotent element and $\phi\colon \mathrm{SL}_2 \mapsto G$ be a homomorphism such that $\phi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = u$. Let $S_\phi = \mathrm{Cent}(\mathrm{im}\,\phi, G) \subset S_u = \mathrm{Cent}(u, G)$ and U^u be the unipotent radical of S_u. Then*

(1) $S_u = S_\phi \cdot U^u$, a semi-direct product. S_ϕ is reductive.

(2) The inclusion $S_\phi \subset S_u$ induces an isomorphism between $S_\phi / S_\phi^\circ Z_G$ and $S_u / S_u^\circ Z_G$.

Let F be a number field and let W_F be the global Weil group of F. For G a split group of type G_2, we can take the dual group $G^* = G_2(\mathbb{C})$. An Arthur parameter is a homomorphism

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \mapsto G^*,$$

defined modulo conjugacy, with the following properties: (The usual definition of Arthur parameter uses Langlands' hypothetical group L_F. But since we are only dealing with principal series, W_F is enough.)

(1) $\psi(W_F)$ is bounded and included in the set of semi-simple elements of G^*.

(2) The restriction of ψ to $\mathrm{SL}_2(\mathbb{C})$ is algebraic.

Let $S_\psi = \mathrm{Cent}(\mathrm{im}\,\psi, G^*)$ and

$$C_\psi = S_\psi / S_\psi^\circ Z_{G^*}.$$

Now we recall Moeglin's reformulation of Arthur's conjecture ([M3]): For each place v of F, we have local Arthur parameters $\psi_v = \psi|W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$, as well as S_{ψ_v}, C_{ψ_v}. It is a part of local Arthur's conjecture that for each irreducible character η_v of C_{ψ_v}, there exists an irreducible representation $\pi(\psi_v, \eta_v)$. For each v, let Π_{ψ_v} be the set of $\pi(\psi_v, \eta_v)$.

We define the global Arthur packet $\Pi_\psi = L^2\big(G(F)\backslash G(\mathbb{A})\big)_\psi$ to be the set of irreducible representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that for each v, π_v belongs to Π_{ψ_v} and for almost all v, π_v is spherical.

ARTHUR'S CONJECTURE (GLOBAL). (1) The representations in the packet corresponding to ψ may occur in the discrete spectrum if and only if C_ψ is finite, *i.e.*, $S_\psi^\circ = 1$. We call such an Arthur parameter elliptic.

(2) For an elliptic Arthur parameter ψ, any $\pi \in \Pi_\psi$ occurs discretely in $L^2\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if

$$\sum_{x \in C_\psi} \prod_v \eta_v(x_v) \neq 0, \tag{3.4}$$

where $\pi = \otimes_v \pi(\psi_v, \eta_v)$, $x = (x_v)$.

REMARK 3.7.1. If C_ψ is abelian, then the above condition is equivalent to: the character $\Pi_v \eta_v|_{C_\psi}$ of C_ψ is trivial. This is what happens in split classical groups [M1]. However, it is not true in our G_2 case since C_ψ is not abelian as we see below. It is S_3, the symmetric group on 3 letters.

Let Π_{res_v} be the subset of Π_{ψ_v}, parametrizing the local components of the residual spectrum. We will find Π_{res_v} and verify (3.4) for a representation $\pi = \otimes_v \pi_v, \pi_v \in \Pi_{\text{res}_v}$ for all v, π_v spherical for almost all v.

REMARK 3.7.2. A representation in Π_{ψ_v} but not in Π_{res_v} will appear as a local component of a cuspidal automorphic representation. Suppose we know the local packet Π_{ψ_v} completely. Then it is a very difficult problem to determine when a representation $\pi \in \Pi_\psi$ is a cuspidal representation. Moeglin [M5] has a partial result on that in the case of split classical groups.

3.7.1. χ trivial.

The Arthur parameter is given by

$$\psi: W_F \times \mathrm{SL}_2(\mathbb{C}) \longmapsto G_2(\mathbb{C}),$$

where $\psi|_{W_F} \equiv 1$ and $\psi: \mathrm{SL}_2(\mathbb{C}) \longmapsto G_2(\mathbb{C})$ is given by a unipotent orbit of $G_2(\mathbb{C})$. In order that ψ be elliptic, the unipotent orbit has to be distinguished. There are two distinguished unipotent orbits of $G_2(\mathbb{C})$, namely, $G_2(\mathbb{C})$ and $G_2(a_1)$ ([Ca, p401]).

CASE 1. The unipotent orbit $G_2(\mathbb{C})$.

The unipotent orbit $G_2(\mathbb{C})$ gives the constant which corresponds to the residue

$$\mathrm{Res}_{\rho_B} \mathrm{Res}_{S_6} A(f, f'; \Lambda).$$

CASE 2. The unipotent orbit $G_2(a_1)$.

If ψ is determined by the unipotent orbit $G_2(a_1)$, then $C_\psi = C_{\psi_v} = S_3$, the symmetric group on 3 letters. There are 3 irreducible characters of S_3, namely, ψ_3, ψ_{21} and ψ_{111}. Here ψ_{111} is the sign character of S_3. They are class functions and the character table is given by

	ψ_3	ψ_{21}	ψ_{111}
C_1	1	2	1
C_2	1	0	-1
C_3	1	-1	1

Character table of S_3

Here C_1, C_2 and C_3 are the conjugacy classes in S_3, namely, $C_1 = \{1\}$, $C_2 = \{(1,2),(1,3),(2,3)\}$, $C_3 = \{(1,2,3),(1,3,2)\}$. From Section 3.6.1, we know that $R_v(\rho_2, \beta_2)I(\beta_2) = \pi_{1v} \oplus \pi_{2v}$, where π_{1v} is spherical. We attach π_{1v} to ψ_3 and π_{2v} to ψ_{21}. Therefore, in this case, $\Pi_{\text{res}_v} = \{\pi_{1v}, \pi_{2v}\}$.

Then $\pi = \otimes_{v\notin S}\pi_{1v} \otimes \otimes_{v\in S}\pi_{2v}$ appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if

$$\otimes_{v\notin S}\psi_3(1) \otimes \bigotimes_{v\in S}\psi_{21}(1) + 2\bigotimes_{v\notin S}\psi_3\big((1,2,3)\big) \otimes \bigotimes_{v\in S}\psi_{21}\big((1,2,3)\big) = 2^s + 2(-1)^s \neq 0, \tag{3.5}$$

i.e., $s \neq 1$, where $s = |S|$. This coincides with Moeglin-Waldspurger [M-W1, Appendix III]: In Moeglin-Waldspurger, there is an operator E_v which acts on π_{1v} and π_{2v} (See Section 3.6.1). Since we attached π_{1v} to ψ_3 and π_{2v} to ψ_{21}, E_v acts on the irreducible characters of S_3 as follows: $E_v(\psi_3) = 1$ and $E_v(\psi_{21}) = -2$. Then we can see that $E_v(\eta) = \eta(1)\eta\big((123)\big)$ for $\eta = \psi_3, \psi_{21}$. Therefore we can write (3.5) as follows:

$$2\bigotimes_{v\notin S}\psi_3\big((1,2,3)\big) \otimes \bigotimes_{v\in S}\psi_{21}\big((1,2,3)\big)\Big(1 + \frac{1}{2}\bigotimes_{v\notin S}\bigotimes_{v\notin S}E_v(\psi_3) \otimes \bigotimes_{v\in S}E_v(\psi_{21})\Big) \neq 0$$

i.e., $\pi = \otimes_v \pi_v$ appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $(1+\frac{1}{2}E)\pi \neq 0$, where $E = \otimes E_v$.

REMARK 3.7.3. According to Arthur's local conjecture, the sign character ψ_{111} should give an irreducible representation which is a local component of a cuspidal automorphic representation. We do not know what it is.

REMARK 3.7.4. For O a distinguished unipotent orbit, let $A(u) = C(u)/C(u)^0$, where $u \in O$ and $C(u)$ is the centralizer of u. Let Springer(O) be the set of irreducible characters of $A(u)$ which are in the image of the Springer correspondence which is an injective map from the set of irreducible characters of W into the set of pairs (O, η), where O is a unipotent orbit and η is an irreducible character of $A(u) = C(u)/C(u)^0$, where $u \in O$. We note that by [Ca, p427], Springer$\big(G_2(a_1)\big) = \{\psi_3, \psi_{21}\}$ in $G_2(\mathbb{C})$. Therefore the local component Π_{res_v} of the residual spectrum attached to the trivial character of the torus is parametrized by Springer$\big(G_2(a_1)\big)$. Moeglin [M1] showed that for split classical groups, the residual spectrum attached to the trivial character of the torus is parametrized by distinguished unipotent orbits O and Springer(O). In other words, if the Arthur parameter ψ is given by the distinguished unipotent orbit O, then $\Pi_{\text{res}_v} =$ Springer(O) and the multiplicity formula (3.4) holds.

Therefore we believe that the same thing would happen for all split groups. We state this as follows:

CONJECTURE. Let G be a split group over a number field F and T be a maximal torus of G. Then the residual spectrum attached to the trivial character of $T(\mathbb{A})/T(F)$ is parametrized by distinguished unipotent orbits O of $G^*(\mathbb{C})$, the L-group of G and Springer(O). More precisely, if the Arthur parameter $\psi\colon \mathrm{SL}_2(\mathbb{C}) \longmapsto G^*(\mathbb{C})$ is given by the distinguished unipotent orbit O, then $\Pi_{\text{res}_v} =$ Springer(O) and the multiplicity formula (3.4) holds.

We give an example of this conjecture in the case of split exceptional group F_4 and we hope to settle this example in the near future: Suppose the Arthur parameter $\psi\colon \mathrm{SL}_2(\mathbb{C}) \longmapsto F_4(\mathbb{C})$ is given by the distinguished unipotent orbit $F_4(a_3)$. By [Ca, p401], $A(u) = S_4$, the symmetric group on 4 letters. There are 5 irreducible characters of S_4, namely, $\psi_4 = 1$, ψ_{31}, ψ_{22}, ψ_{211} and ψ_{1111}. By [Ca, p428], Springer$\big(F_4(a_3)\big) = \{\psi_4, \psi_{31}, \psi_{22}, \psi_{211}\}$. The character table of S_4 is given by

	C_1	C_2	C_3	C_4	C_5
ψ_4	1	1	1	1	1
ψ_{1111}	1	1	1	-1	-1
ψ_{211}	2	2	-1	0	0
ψ_{22}	3	-1	0	1	-1
ψ_{31}	3	-1	0	-1	1

Character table of S_4

Here $C_i, i = 1, \ldots, 5$, are the conjugacy classes in S_4, with representatives 1, (12)(34), (123), (12) and (1234), respectively: $|C_1| = 1, |C_2| = 3, |C_3| = 8, |C_4| = 6$ and $|C_5| = 6$. According to the conjecture, there will be 4 irreducible representations $\pi_{1v}, \ldots, \pi_{4v}$ attached to ψ_4, ψ_{211}, ψ_{22} and ψ_{31}, respectively. We divide Springer$\big(F_4(a_3)\big)$ as follows: Springer$\big(F_4(a_3)\big) = \Pi_1 \cup \Pi_2 \cup \Pi_3$, where $\Pi_1 = \{\psi_4, \psi_{211}\}, \Pi_2 = \{\psi_4, \psi_{22}\}$ and $\Pi_3 = \{\psi_4, \psi_{31}\}$. The residual spectrum factors through Π_i, *i.e.*, it is the set of all $\pi = \otimes \pi_v$ such that there exists i, $\pi_v \in \Pi_i$ for all v. Let S be a finite set of finite places and $s = |S|$. If $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$, it appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $2^s + 3(2^s) + 8(-1)^s \neq 0$, *i.e.*, $s \neq 1$. If $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{3v}$, it appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $3^s + 3(-1)^s + 6 + 6(-1)^s \neq 0$, *i.e.*, $s \neq 1$. If $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{4v}$, it appears in $L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)$ if and only if $3^s + 3(-1)^s + 6(-1)^s + 6 \neq 0$, *i.e.*, $s \neq 1$.

3.7.2. χ non-trivial. In order to find Arthur parameters for non-trivial characters, we have to look for endoscopic groups of $G_2(\mathbb{C})$, since Arthur parameters will factor through the endoscopic groups.

There are two equivalence classes of proper cuspidal endoscopic groups of $G_2(\mathbb{C})$, that is, $SL_3(\mathbb{C})$ and $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$ ([A1, p30]). They are given as follows: Under the identification $M_1 \simeq GL_2$, by (2.1), $\beta_4^\vee(t) = \text{diag}(t, t)$. Then by [Ca, p93], $C\big(\beta_4^\vee(\omega)\big)$, the centralizer of $\beta_4^\vee(\omega)$ in $G_2(\mathbb{C})$, where $\omega^3 = 1, \omega \neq 1$, is reductive and its root system is $\Phi_1 = \{\pm\beta_1, \pm\beta_3, \pm\beta_5\}$, *i.e.*, $C\big(\beta_4^\vee(\omega)\big) \simeq SL_3$. The other one is $C\big(\beta_4^\vee(-1)\big)$. By [Ca, p93], its root system is $\Phi_1 = \{\pm\beta_1, \pm\beta_4\}$, *i.e.*, $C\big(\beta_4^\vee(-1)\big) \simeq SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$.

The center of $SL_3(\mathbb{C})$ is $Z_3 = \{\omega I_3, \omega^3 = 1\} \simeq \mathbb{Z}_3$ and the center of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})/\{\pm 1\}$ is $Z_2 = \{\pm I_2\}$. Moreover, $S_3 = Z_3 \rtimes Z_2$.

CASE 1. The conjugacy class of χ: $\chi_1 = 1, \chi_6^2 = 1, \chi_6 \neq 1$

Under the identification $M_1 \simeq GL_2$, $\chi = \chi(\mu, \mu)$, $\mu^2 = 1$, $\mu \neq 1$, where μ is a grössencharacter of F. We have the embedding $SL_3(\mathbb{C}) \subset G_2(\mathbb{C})$.

The Arthur parameter factors through $SL_3(\mathbb{C})$:

$$\psi\colon W_F \times SL_2(\mathbb{C}) \longmapsto SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

$\psi|_{W_F}\colon w \longmapsto \begin{pmatrix} \mu(w) & & \\ & \mu(w) & \\ & & 1 \end{pmatrix}$ and $\psi\colon SL_2(\mathbb{C}) \longmapsto SL_3(\mathbb{C})$ is determined by the principal unipotent orbit of $SL_3(\mathbb{C})$. Here we note that under the embedding $SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$, the principal unipotent orbit of $\mathrm{SL}_3(\mathbf{C})$ corresponds to the distinguished unipotent orbit $G_2(a_1)$ in $G_2(\mathbf{C})$ ([Ca, p401]). Then $S_\psi = Z_2$, $C_\psi = Z_2$ and $C_{\psi_v} = Z_2$ if μ_v is not trivial. $C_{\psi_v} = S_3$ if μ_v is trivial.

If μ_v is trivial, then $\Pi_{\mathrm{res}_v} = \{\pi_{1v}, \pi_{2v}\}$. If μ_v is not trivial, then $\Pi_{\mathrm{res}_v} = \{\pi_{1v}\}$. Then any $\pi = \otimes_{v \notin S} \pi_{1v} \otimes \otimes_{v \in S} \pi_{2v}$ appears in $L^2_d\big(G(F)\backslash G(\mathbb{A})\big)$ since

$$\bigotimes_{v\notin S} \psi_3(1) \otimes \bigotimes_{v\in S} \psi_{21}(1) + \bigotimes_{v\notin S} \psi_3(\tau) \otimes \bigotimes_{v\in S} \psi_{21}(\tau) \neq 0,$$

where τ is the non-trivial element in Z_2.

CASE 2. The conjugacy class of $\tilde{\chi}$: $\tilde{\chi}_1^3 = 1, \tilde{\chi}_1 \neq 1, \tilde{\chi}_6 = 1$

Under the identification, $M_1 \simeq \mathrm{GL}_2$, $\tilde{\chi} = \chi(\mu, \nu)$, $\mu^3 = 1, \mu \neq 1, \nu = \mu^2$. The Arthur parameter factors through $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})/\{\pm 1\}$:

$$\psi\colon W_F \times \mathrm{SL}_2(\mathbf{C}) \longmapsto \mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})/\{\pm 1\} \hookrightarrow G_2(\mathbf{C}),$$

where $\psi|_{W_F}\colon w \longmapsto \begin{pmatrix} \mu(w) & \\ & \mu^{-1}(w) \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\psi\colon \mathrm{SL}_2(\mathbf{C}) \longmapsto \mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})$ is determined by the principal unipotent orbits of $\mathrm{SL}_2(\mathbf{C})$. We note that under the embedding $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})/\{\pm 1\} \hookrightarrow G_2(\mathbf{C})$, the principal unipotent orbit of $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})/\{\pm 1\}$ corresponds to the unipotent orbit A_1 in $G_2(\mathbf{C})$ ([Ca, p401]). Then $S_\psi = Z_2$, $C_{\psi_v} = Z_2$ for μ_v non-trivial and $C_{\psi_v} = 1$ for μ_v trivial. In this case, Π_{res_v} consists of the Langlands' quotient which corresponds to the trivial character of $C_{\psi_v} = Z_2$. Therefore Π_{res} consists of one element.

REMARK 3.7.5. Arthur associated to an Arthur parameter ψ, an associated Langlands' parameter ϕ_ψ and conjectured that we could enlarge the L-packet $\Pi_{\phi_{\psi_v}}$ to Π_{ψ_v}. We note that in each of our cases, the associated Langlands' L-packet consists of only one element.

4. Decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$. We have

$$\mathfrak{a}^*_{P_1} = X(M_1) \otimes \mathbf{R} = \mathbf{R}\beta_4, \mathfrak{a}_{P_1} = \mathbf{R}\beta_4^\vee$$

ρ_{P_1} is the half sum of roots generating N_1. Then $\rho_{P_1} = \frac{5}{2}\beta_4$.

Let $\tilde{\alpha} = \beta_4$ and identify $s \in \mathbf{C}$ with $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbf{C}}$. Let $\pi = \otimes \pi_v$ be a cusp form on $M_1 = \mathrm{GL}_2$. Given a K-finite function φ in the space of π, we shall extend φ to a function $\tilde{\varphi}$ on G and set

$$\Phi_s(g) = \tilde{\varphi}(g) \exp\langle s + \rho_{P_1}, H_{P_1}(g)\rangle.$$

Define an Eisenstein series

$$E(s, \tilde{g}, g, \rho_1) = \sum_{\gamma \in P_1(F)\backslash G(F)} \Phi_s(\gamma g).$$

It is known that $E(s, \tilde{\varphi}, g, \rho_1)$ converges for $\mathrm{Re}(s) >> 0$ and extends to a meromorphic function of s in $\mathbf{C}$, with a finite number of poles in the plane $\mathrm{Re}(s) > 0$, all simple and on the real axis.

It is also known that $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$ is spanned by the residues of the Eisenstein series for $\mathrm{Re}(s) > 0$. We know that the poles of the Eisenstein series coincide with those of its constant terms. So it is enough to consider the constant term along P_1, which is

$$E_0(s, \tilde{\varphi}, g, P_1) = \sum_{w\in\Omega} M(s, \pi, w)f(g)$$

where $\Omega = \{1, \rho_6\rho_1\rho_6\rho_1\rho_6\}$ and

$$M(s, \pi, w)f(g) = \int_{N_w^-} f(w^{-1}ng)\,dn$$

where

$$N_w^- = \prod_{\substack{\alpha>0\\ w^{-1}\alpha<0}} U_\alpha,$$

U_α is the one parameter unipotent subgroup and $f \in I(s, \pi) = \mathrm{Ind}_{P_1}^G\, \pi \otimes \exp\big(s, H_{P_1}()\big)$.

We note that for each s, the representation of $G(\mathbb{A})$ on the space of Φ_s is equivalent to $I(s, \pi)$.

Then

$$M(s, \pi, w) = \bigotimes M(s, \pi_v, w), \quad M(s, \pi_v, w)f_v(g) = \int_{N_w(F_v)} f_v(w^{-1}ng)\,dn$$

where $f = \otimes f_v, f_v$ is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ for almost all v.

Let ${}^LM_1 = \mathrm{GL}_2(\mathbb{C})$ be the L-group of M_1. Denote by r the adjoint action of ${}^LM_1$ on the Lie algebra ${}^L\mathfrak{n}_1$ of ${}^LN_1$, the L-group of N_1.

Then

$$r = r_1 \oplus r_2$$
$$r_1 = r_3^0, \quad r_2 = \wedge^2\rho_2$$

where $r_3^0 = r_3 \otimes (\wedge^2\rho_2)^{-1}$ is the adjoint cube representation of $\mathrm{GL}_2(\mathbb{C})$ (See [S4]). Here r_3 is the symmetric cube representation of $\mathrm{GL}_2(\mathbb{C})$ and ρ_2 is the standard representation of $\mathrm{GL}_2(\mathbb{C})$.

Then it is well-known ([S3]) that for $w = \rho_6\rho_1\rho_6\rho_1\rho_6$

$$M(s, \pi, w)f = \bigotimes_{v\in S} M(s, \pi_v, w)f_v \otimes \bigotimes_{v\notin S} \longrightarrow \tilde{f}_v \times \frac{L_S(s, \pi, r_1)L_S(2s, \pi, r_2)}{L_S(1+s, \pi, r_1)L_S(1+2s, \pi, r_2)}$$

where S is a finite set of places of F, including all the archimedean places such that for every $v \notin S$, π_v is a class 1 representation and if $f = \otimes_v f_v$, for $v \notin S, f_v$ is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. $\tilde{f}_v$ is the K_v-fixed function in the space of $I\big(-s, w(\pi_v)\big)$.

Finally, $L_S(s, \pi, r_i) = \Pi_{v\notin S} L(s, \pi_v, r_i)$, where $L(s, \pi_v, r_i)$ is the local Langlands' L-function attached to π_v, r_i.

(1) Analysis of $L_S(s, \pi, r_1)$

We know ([S4]) that $L_S(s,\pi,r_1)$ is absolutely convergent for $\mathrm{Re}(s) > 1$ and hence has no zero there. It is expected ([S4, Bu-G-H, Ik]) that the completed L-function $L(s,\pi,r_1)$ has a pole for $\mathrm{Re}(s) > 0$ if and only if $s = 1$, $\omega_\pi^2 = 1$, $\omega_\pi \neq 1$ and π is the monomial representation corresponding to the quadratic character ω_π, where ω_π is the central character of π. We assume this fact.

REMARK. Ikeda [Ik] calculated the poles of the Rankin triple L-function $L(s,\pi\otimes\pi\otimes\pi)$ for π cuspidal representation of GL_2. It is related to the symmetric cube L-function of π as follows:

$$L(s,\pi\otimes\pi\otimes\pi) = L(s,\pi,r_3)\big(L(s,\pi\otimes\omega_\pi)\big)^2.$$

The symmetric cube L-function is given by

$$L(s,\pi,r_3) = L(s,\pi\otimes\omega_\pi,r_3^0).$$

$L(s,\pi,r_3)$ has a pole at $s = 1$ when $\omega_\pi^6 = 1$ and $\omega_\pi^3 \neq 1$ and so $L(s,\pi,r_3^0)$ has a pole when $\omega_\pi^2 = 1$ and $\omega_\pi \neq 1$.[1]

(2) Analysis of $L_S(s,\pi,r_2)$

For $v \notin S$,

$$L(s,\pi_v,r_2) = L(s,\omega_{\pi_v}) = (1-\omega_{\pi_v}(\varpi)q_v^{-s})^{-1}$$

so $L_S(s,\pi,r_2)$ is the (partial) Hecke L-function. It has no zero for $\mathrm{Re}(s) > 1$. The completed L-function $L(s,\pi,r_2)$ has a pole for $\mathrm{Re}(s) > 0$ if and only if $s = 1$, $\omega_\pi = 1$.

(3) Analysis of $M(s,\pi_v,w)$ for $v \in S$.

For π_v tempered, the local factors $L(s,\pi_v,r_i)$ and $M(s,\pi_v,w)$ are holomorphic for $\mathrm{Re}(s) > 0$. We show that for any $v \in S$,

$$L(s,\pi_v,r_1)^{-1}L(2s,\pi_v,r_2)^{-1}M(s,\pi_v,w)$$

is holomorphic. It is enough to show it for π_v complementary series. We follow [Ki]. Under the identification $M_1 \simeq \mathrm{GL}_2$, by (2.1), for $\pi_v = \pi(\mu|\ |^r,\mu|\ |^{-r})$, $0 \leq r < \frac{1}{2}$, complementary series of GL_2,

$$\mathrm{Ind}_{P_1}^G \pi_v \otimes \exp\big(\langle s\tilde{\alpha}, H_{P_1}()\rangle\big) = \mathrm{Ind}_B^G \chi(\mu,\mu)\otimes\exp\big(\langle\Lambda, H_B()\rangle\big),$$

where $\Lambda = (2r)\beta_3 + (s-3r)\beta_4$. From this we have our assertion.

Now we assume that $r < \frac{1}{6}$. Right now the best known result is $r < \frac{1}{5}$ due to Shahidi [S3]. Then Λ is in the positive Weyl chamber for $s = \frac{1}{2}$ and $s = 1$ and we have

LEMMA 4.1. *For each v, the images of $M(\frac{1}{2},\pi_v,w)$ and $M(1,\pi_v,w)$ are irreducible.*

---

[1] Thanks to F. Shahidi who pointed this out.

Conclusion. $E(s, \tilde{\varphi}, g, P_1)$ has a pole in the half plane $\mathrm{Re}(s) > 0$ if and only if

(1) $\omega_\pi = 1, s = \frac{1}{2}, L(\frac{1}{2}, \pi, r_3^0) \neq 0$,

(2) $\omega_\pi^2 = 1, \omega_\pi \neq 1, s = 1$, π monomial representation attached to ω_π.

Let $J_1(\pi_v)$ be the image of $M(\frac{1}{2}, \pi_v, w)$ and $J_2(\pi_v)$, the image of $M(1, \pi_v, w)$. They are the unique irreducible quotients of $I(\frac{1}{2}, \pi_v)$ and $I(1, \pi_v)$, respectively. Let $J_1(\pi) = \otimes_v J_1(\pi_v)$ and $J_2(\pi) = \otimes_v J_2(\pi_v)$. We have proved

THEOREM 4.2.

$$L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1} = \bigoplus_{\pi_1} J_1(\pi_1) \oplus \bigoplus_{\pi_2} J_2(\pi_2)$$

where π_1 *runs over cuspidal representations of* GL_2 *with trivial central characters and* $L(\frac{1}{2}, \pi, r_3^0) \neq 0$ *and* π_2 *runs over monomial representations.*

5. Decomposition of $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2}$.

In this case $\mathfrak{a}^*_{P_2} = X(M_2) \otimes \mathbb{R} = \mathbb{R}\beta_3$, $\mathfrak{a}_{P_2} = \mathbb{R}\beta_3^\vee$, $\rho_{P_2} = \frac{3}{2}\beta_3$.

Let $\tilde{\alpha} = \beta_3$ and identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$. In this case, for π cuspidal representation of GL_2, the constant term of Eisenstein series is given by

$$E_0(s, \tilde{\varphi}, g, P_2) = \sum_{w \in \Omega} M(s, \pi, w) f(g)$$

where $\Omega = \{1, \rho_1\rho_6\rho_1\rho_6\rho_1\}$.

The adjoint action r of ${}^L M_2$ on ${}^L\mathfrak{n}_2$ is given as

$$r = r_1 \oplus r_2 \oplus r_3$$
$$r_1 = \rho_2, r_2 = \wedge^2\rho_2, r_3 = \rho_2 \otimes \wedge^2\rho_2.$$

Therefore for $w = \rho_1\rho_6\rho_1\rho_6\rho_1$,

$$M(s, \pi, w) f = \bigotimes_{v \in S} M(s, \pi_v, w) f_v \otimes \bigotimes_{v \notin S} \tilde{f}_v \times \frac{L_S(s, \pi, r_1) L_S(2s, \pi, r_2) L_S(3s, \pi, r_3)}{L_S(s+1, \pi, r_1) L_S(2s+1, \pi, r_2) L_S(3s+1, \pi, r_3)}$$

where S is the same as in the case $L^2_{\mathrm{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_1}$.

Here

$$L(s, \pi_v, r_1) = L(s, \pi_v), \text{ the standard } L\text{-function for } \mathrm{GL}_2.$$
$$L(s, \pi_v, r_3) = L(s, \pi_v \otimes \omega_{\pi_v}), \text{ twisted by the central character.}$$
$$L(s, \pi_v, r_2) = L(s, \omega_{\pi_v}), \text{ Hecke } L\text{-function.}$$

We know that $L_S(s, \pi \otimes \theta)$ is absolutely convergent for $\mathrm{Re}s > 1$ for any grössencharacter θ. So it has no zero there. We know also that the completed L-function $L(s, \pi \otimes \theta)$ is entire for any θ. Under the identification $M_2 \simeq \mathrm{GL}_2$, by (2.2), for $\pi_v = \pi(\mu|\ |^r, \mu|\ |^{-r})$ complementary series of GL_2,

$$\mathrm{Ind}_{P_2}^G \pi_v \otimes \exp\big(\langle s\tilde{\alpha}, H_{P_2}()\rangle\big) = \mathrm{Ind}_B^G \chi(\mu, \mu) \otimes \exp\big(\langle \Lambda, H_B()\rangle\big),$$

where $\Lambda = (s - 3r)\beta_3 + 6r\beta_4$. Therefore, for any $v \in S$,

$$\prod_{i=1}^{3} L(is, \pi_v, r_i)^{-1} M(s, \pi_v, w)$$

is holomorphic. Also if we assume $r < \frac{1}{6}$, Λ is in the positive Weyl chamber and the image of $M(\frac{1}{2}, \pi_v, w)$ is irreducible. Therefore, $E(s, \tilde{\varphi}, g, P_2)$ has a pole in the half plane $Res > 0$ if and only if $\omega_\pi = 1, s = \frac{1}{2}$ and $L(\frac{1}{2}, \pi, r_1) \neq 0$.

Let $J(\pi_v)$ be the image of $M(\frac{1}{2}, \pi_v, w)$ and $J(\pi) = \otimes_v J(\pi_v)$. Then we have

THEOREM 5.1.
$$L^2_{\text{dis}}\big(G(F)\backslash G(\mathbb{A})\big)_{M_2} = \bigoplus_{\pi} J(\pi)$$
where π runs over cuspidal representations of GL_2 *with trivial central characters and* $L(\frac{1}{2}, \pi) \neq 0$.

REMARK 5.1. The referee suggested the problem of finding a connection between Arthur's conjecture and the non-vanishing of L-functions at $s = \frac{1}{2}$. Arthur [A1] did it for the group $P\,\mathrm{Sp}_4$. It would be interesting to do so in the above case.

REFERENCES

[A1] J. Arthur, *On some problems suggested by the trace formula*, In: Lie group representations, II, Lecture Notes in Math. **1041**, Springer-Verlag, 1984, 1–49.

[A2] ———, *Unipotent Automorphic Representations: Conjectures*, In: Orbites unipotentes et représentations II. Groupes p-adiques et réels, Astérisque **171–172**(1989), 13–71.

[A3] ———, *Eisenstein Series and the Trace Formula*, Proceedings of Symposia in Pure Mathematics, 1, **33**(1979), 253–274.

[B-V] D. Barbasch and D. Vogan, Jr., *Unipotent representations of complex semisimple groups*, Ann. of Math. **121**(1985), 41–110.

[Bu-G-H] D. Bump, D. Ginzburg and J. Hoffstein, *The Symmetric Cube*, 1995, preprint.

[Ca] R. W. Carter, *Finite Groups of Lie Type, Conjugacy Classes and Complex Characters*, John Wiley and Sons, 1985.

[G-S] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-functions*, Perspectives in Mathematics **6**, Academic Press, 1988.

[Go] L. Goldstein, *Analytic Number Theory*, Prentice-Hall, 1971.

[Ik] T. Ikeda, *On the location of poles of the triple L-functions*, Comp. Math. **83**(1992), 187–237.

[Ja] H. Jacquet, *On the residual spectrum of* $\mathrm{GL}(n)$, In: Lie Group Representations II, Lecture Notes in Math. **1041**, Springer-Verlag, 1983, 185–208.

[K-S] C. D. Keys and F. Shahidi, *Artin L-functions and normalization of intertwining operators*, Ann. Scient. Ec. Norm. Sup. **21**(1988), 67–89.

[Ki] H. Kim, *The Residual Spectrum of* Sp_4, Comp. Math. (2)**99**(1995), 129–151.

[Ki-S] H. Kim and F. Shahidi, *Quadratic unipotent Arthur parameters and residual spectrum of* Sp_{2n}, Amer. J. Math. (2)**118**, 401–425.

[L1] R. P. Langlands, *Representations of abelian algebraic groups*, Notes, Yale University, 1968.

[L2] ———, *On the classification of irreducible representations of real algebraic groups*, In: Representation Theory and Harmonic Analysis on Semisimple Lie Groups, (eds. Paul J. Sally and David A. Vogan), AMS Math. Surveys and Monographs **31**(1989), 101–170.

[L3] ———, *Euler Products*, Yale University Press, 1971.

[L4] ———, *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Math. **544**, Springer-Verlag, 1976.

[Li-Sc] J. S. Li and J. Schwermer, *Constructions of automorphic forms and related cohomology classes for arithmetic subgroups of* G_2, Comp. Math. **87**(1993), 45–78.

[M1] C. Moeglin, *Représentations unipotentes et formes automorphes de carréintégrable*, Forum mathematicum **6**(1994), 651–744.

[M2] ______, *Orbites unipotentes et spectre discret non ramifie, Le cas des groupes classiques déployé*, Comp. Math. 77(1991), 1–54.

[M3] ______, *Une conjecture sur le spectre résiduel des groupes classiques*, 1994, preprint.

[M4] ______, *Letter*, 1994.

[M5] ______, *Cuspidal quadratic unipotent representations*, 1994, preprint.

[M-W1] C. Moeglin and J. L. Waldspurger, *Spectral decomposition and Eisenstein series, Une Paraphrase de l'Écriture*, Cambridge University Press, 1995.

[M-W2] ______, *Le Spectre Résiduel de* GL(n), Ann. Scient. Éc. Norm. Sup. **22**(1989), 605–674.

[R] F. Rodier, *Decomposition de la serie principale des groupes redutifs p-adiques*, In: Non-commutative harmonic analysis, Lecture Notes in Math. **880**, Springer-Verlag, 1981, 408–424.

[S1] F. Shahidi, *On certain L-functions*, Amer. J.Math. **103**(1981), 297–356.

[S2] ______, *Whittaker Models for Real Groups*, Duke Math. J. (1)**47**(1980), 99–125.

[S3] ______, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math. **127**(1988), 547–584.

[S4] ______, *Third symmetric power L-functions for* GL(2), Comp. Math. **70**(1989), 245–273.

[Sp] N. Spaltenstein, *Classes Unipotentes et Sous-groupes de Borel*, Lecture Notes in Mathematics **946**, Springer-Verlag, 1982.

[Wi] N. Winarsky, *Reducibility of principal series representations of p-adic Chevalley groups*, Amer. J. Math. (5)**100**(1978), 941–956.

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COMPACTNESS OF A LOCALLY COMPACT GROUP G AND GEOMETRIC PROPERTIES OF $A_p(G)$

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ABSTRACT. Let G be a locally compact topological group. A number of characterizations are given of the class of compact groups in terms of the geometric properties such as Radon-Nikodym property, Dunford-Pettis property and Schur property of $A_p(G)$, and the properties of the multiplication operator on $PF_p(G)$. We extend and improve several results of Lau and Ülger [17] to $A_p(G)$ and $B_p(G)$ for arbitrary p.

1. **Introduction.** Let G be a locally compact topological group. Let $A(G)$ and $B(G)$ be the Fourier and the Fourier-Stieltjes algebras of G, respectively. Taylor [22] showed that $A(G)$ has the Radon-Nikodym property if and only if $G \in [AR]$ and $B(G)$ has the Radon-Nikodym property if and only if $G \in [AU]$. Later, Lau and Ülger [17] studied extensively the geometric properties of $A(G)$ and $B(G)$. The main machinery for them is C^*-algebraic since, as is well known, the predual of $B(G)$ is the group C^*-algebra $C^*(G)$ and the dual of $A(G)$ is the von Neumann algebra $VN(G)$. In this paper, we will establish several sufficient conditions for G to be compact. We point out that the C^*-algebraic method, which worked for $p = 2$, does not work for arbitrary p since $A_p(G)$ and $B_p(G)$ are usually not related to any C^*-algebra. As we know, $A_2(G) = A(G)$ for an arbitrary locally compact group G, and if G is amenable then $B_2(G) = B(G)$ and $PF_2(G) = C^*(G)$, the group C^*-algebra of G. We extend and improve several results of Lau and Ülger [17] to $A_p(G)$ and $B_p(G)$ for $1 < p < \infty$ (see Lau and Ülger [17], page 321–322). We prove in this paper, among other things, the following: for any locally compact group G, the following are equivalent

(a) G is compact;
(b) G is amenable and each functional ϕ in $B_p(G)$ is almost periodic on $PF_p(G)$;
(c) Each functional ϕ in $A_p(G)$ is almost periodic on $PF_p(G)$;
(d) There exists a nonzero functional ϕ in $A_p(G)$ which is almost periodic on $PF_p(G)$;
(e) For each f in $PF_p(G)$, the multiplication operator $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ (defined by $\tau_f(g) = f \cdot g$ for all $g \in PF_p(G)$) is compact;
(f) G is amenable and for each f in $PF_p(G)$, the multiplication operator $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is weakly compact;

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- (g) G is amenable and the Banach algebra $PF_p(G)$ is an ideal of its second dual under either Arens product;
- (h) For any $a \in A_p(G)$, or equivalently for a nonzero $a \in A_p(G)$, $\{_x a : x \in G\}$ is a compact subset of $A_p(G)$.

We also prove that each of the following conditions is sufficient for G to be compact:

- (i) The space $A_p(G)$ has the Schur property;
- (j) G is amenable and the space $B_p(G)$ has *RNP* and *DPP*;
- (k) The algebra $PF_p(G)$ does not contain an isomorphic copy of ℓ^1 and has the *DPP*;
- (l) Every bounded linear operator $T: PF_p(G) \longrightarrow B_p(G)$ is compact.

The main tools we use to prove our results are Lemma 3.1, Lemma 4.5 and Lemma 5.1 in which various sequences approaching "infinity" will be considered. We organize this paper as follows. In Section 2, we collect the notations and definitions that will be used in this paper. In Section 3 we give various sufficient conditions for a locally compact group G to be compact in terms of some geometric properties of $A_p(G)$. In Section 4, by developing the technique used in Section 3 we characterize the compact groups in terms of almost periodic functionals on $PF_p(G)$ and $PM_p(G)$. In Section 5 we give some results relating the properties of the multiplication operator $\tau_f: PF_p(G) \longrightarrow PF_p(G)$ to the properties of the group G.

2. **Preliminaries and Some Notations.** Let G be a locally compact group with a fixed left Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated Lebesgue spaces ($1 \le p \le \infty$). For any $f: G \longrightarrow C$, and $x \in G$, the left [right] translation of f by x is defined by $_x f(y) = f(xy)$ [$f_x(y) = f(yx)$], and $\check{f}$ is defined by $\check{f}(y) = f(y^{-1})$, $y \in G$. We say that G is amenable if there exists an $m \in L^\infty(G)^*$ such that $m \ge 0$, $m(1) = 1$ and $m(_x f) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. Properties of amenable groups can be found in Greenleaf [13], Paterson [20] and Pier [21]. For any two measurable functions f and g on G, their convolution $f * g$ is defined by

$$f * g(x) = \int_G f(t) g(t^{-1}x)\, dt$$

whenever this makes sense. The Figà-Talamanca-Herz algebra $A_p(G)$ is the space of functions $u: G \longrightarrow C$ which can be represented, nonuniquely, as

$$u = \sum_{n=1}^{\infty} v_n * \check{u}_n \text{ for } u_n \in L^p(G) \text{ and } v_n \in L^q(G) \text{ with } \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$$

and $\|u\| = \inf \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q$, where the infimum is taken over all possible representations of u and $\frac{1}{p} + \frac{1}{q} = 1$. It is known that $A_p(G)$ is a regular tauberian algebra under the pointwise multiplication and its Gelfand spectrum is G. Furthermore the algebra $A_p(G) \subset C_0(G)$ and it has a bounded approximate identity if and only if G is amenable. If $p = 2$, $A_p(G) = A(G)$, the Fourier algebra of G. Each $f \in L^1(G)$ defines a bounded linear functional on $A_p(G)$ by

$$\langle f, u \rangle = \int_G f(x) u(x)\, dx \text{ for } u \in A_p(G),$$

and it defines a bounded linear operator on $L^p(G)$ by $f(g) = f * g, g \in L^p(G)$. The norm of f as an element of $A_p(G)^*$ and the operator norm of f as an element of $\mathcal{B}\big(L^p(G)\big)$ are the same, that is,

$$\|f\|_{A_p(G)} = \sup_{u \in A_p(G), \|u\| \leq 1} |\langle f, u \rangle| \text{ and } \|f\|_{\mathcal{B}\left(L^p(G)\right)} = \sup_{g \in L^p(G), \|g\|_p \leq 1} \|f * g\|_p$$

are equal. It follows that $L^1(G)$ is a subspace of $A_p(G)^*$. Let $PF_p(G)$ and $PM_p(G)$ be the closures of $L^1(G)$ in $A_p(G)^*$ with respect to the norm topology and the *weak** topology respectively. It can be shown that $PM_p(G) = A_p(G)^*$. When $p = 2$, $PM_2(G)$ is usually denoted by $VN(G)$ while $PF_2(G)$ is $C^*_\rho(G)$, the reduced group C^*-algebra of G. When G is amenable, $C^*_\rho(G) = C^*(G)$, the group C^*-algebra of G.

$B_p(G)$ is the pointwise multiplier algebra of $A_p(G)$, consisting of the continuous functions v on G such that $uv \in A_p(G)$ for all $u \in A_p(G)$. The norm on $B_p(G)$ is defined by

$$\|v\|_{B_p(G)} = \sup\{\|vu\|_{A_p(G)} \text{ and } \|u\|_{A_p(G)} \leq 1\}.$$

Observe that $A_p(G) \subseteq B_p(G)$ and if $v \in A_p(G)$, $\|v\|_{B_p(G)} \leq \|v\|_{A_p(G)}$. It has been showed that $PF_p(G)^* = B_p(G)$ if and only if G is amenable (see Cowling [5]), in this case, for any $b \in B_p(G)$, $b \in PF_p(G)^*$ is defined by $\langle b, f \rangle = \int_G b(t) f(t)\, dt$ for $f \in L^1(G)$. When $p = 2$, $B_2(G) = B(G)$, the Fourier-Stieltjes algebra of G which is the dual of the group C^*-algebra $C^*(G)$.

For any $f \in PM_p(G)$ and $b \in A_p(G)$ or $b \in B_p(G)$, $f \circ b$ is a functional on $PM_p(G)$ or $PF_p(G)$ defined by $\langle f \circ b, g \rangle = \langle b, g \cdot f \rangle$ whenever this makes sense, where $g \cdot f$ is the product of g and f in $PM_p(G)$. Let $x \in G$, δ_x denotes the point measure at x.

Let X be a Banach space. We recall definitions of some geometric properties of X as follows.

Radon-Nikodym property (RNP). The Banach space X is said to have the RNP if each closed convex subset D of X is dentable, *i.e.* for any $\epsilon > 0$ there exists an $x \in D$ such that $x \notin \overline{Co}\big(D \sim B_\epsilon(x)\big)$, where $B_\epsilon(x) = \{y \in X : \|x - y\| < \epsilon\}$. It is known that X^* has the RNP if and only if every separable subspace of X has a separable dual (see Bourgin [3], Diestel [6] and Huff [16] for more information). Granirer [11] proved that $A_p(G)$ has the RNP if G is compact.

Dunford-Pettis property (DPP). The Banach space X is said to have the DPP if, for any Banach space Y, every weakly compact linear operator $u : X \longrightarrow Y$ sends weakly Cauchy sequences into norm convergent sequences. See Diestel [7] for more information on this property.

Schur property. The Banach space X is said to have the Schur property if every weakly convergent sequence is norm convergent.

Let A be an arbitrary Banach algebra. For any $a \in A$ and $f \in A^*$, $af \in A^*[fa]$ is defined by $\langle af, b \rangle = \langle f, ba \rangle [\langle fa, b \rangle = \langle f, ab \rangle]$. The functional f is said to be *weakly almost periodic* [*almost periodic*] on A if $\{af : a \in A, \|a\| \leq 1\}$ is relatively weakly compact [relatively compact]. By wap(A) [ap(A)] we denote the linear subspace of A^* consisting of all weakly almost periodic [almost periodic] functionals on A.

3. The DPP, the RNP and the Schur property of $A_p(G)$ and $B_p(G)$.

In this section we will be concerned with geometric properties of $A_p(G)$ and $B_p(G)$. A number of sufficient conditions for G to be compact will be established. Our main tool is the following lemma. We give an elementary proof here.

LEMMA 3.1. *Let G be a locally compact group. If G is not compact then there is a sequence $\{x_n\}$ in G such that for any compact set K of G there is an N with $x_n \notin K$ for all $n > N$. Furthermore, for any $a \in A_p(G)$, ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$.*

PROOF. Since G is not compact we can choose an open and closed subgroup G_0 of G which is σ-compact. Let $G_0 = \bigcup_{n=1}^{\infty} K_n$, where K_n is an open set with compact closure and $K_n \subset K_{n+1}$ for all n. For each $n \geq 1$, choose an $x_n \in K_{n+1} \sim K_n$. Then for any compact set K of G we have $K \cap G_0 \subseteq \bigcup_{i=1}^{N} K_i \cap K$ for some natural number N. So $x_n \notin K$ for all $n > N$.

Let $a \in A_p(G)$. We now show that ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$. Let $\epsilon > 0$ and let $F \in PM_p(G)$ be fixed. By the definition of $A_p(G)$ there are $f_n \in L^p(G)$ and $g_n \in L^q(G)$ such that $a = \sum_{n=1}^{\infty} g_n * \check{f}_n$ and $\sum_{n=1}^{\infty} \|g_n\|_q \|f_n\|_p < \infty$. Since continuous functions with compact supports are dense in $L^p(G)$ and $L^q(G)$, there exists an $u = \sum_{n=1}^{k} v_n * \check{u}_n$ in $A_p(G)$ such that $\|u - a\| < \epsilon$, where all u_n and v_n are continuous on G with compact supports. So $\|{}_{x_n}u - {}_{x_n}a\| = \|u - a\| < \epsilon$ for all n and

$$\begin{aligned} |\langle F, {}_{x_n}a\rangle| &\leq |\langle F, {}_{x_n}a - {}_{x_n}u\rangle| + |\langle F, {}_{x_n}u\rangle| \\ &\leq \|F\|\,\|{}_{x_n}a - {}_{x_n}u\| + |\langle F, {}_{x_n}u\rangle| \\ &\leq \epsilon\|F\| + |\langle F, {}_{x_n}u\rangle|. \end{aligned}$$

To prove that $F({}_{x_n}a) \longrightarrow 0$, it suffices to show that $\langle F, {}_{x_n}(v_i * \check{u}_i)\rangle \longrightarrow 0$ for each $1 \leq i \leq k$. Let i be fixed and let V be the support of v_i. Then V is compact and $|v_i(t)| \leq \|v_i\|_\infty 1_V(t)$ for all t. Recall that if T is in $PM_p(G)$ and $v * \check{u}$ is a simple element of $A_p(G)$ then $\langle T, v * \check{u}\rangle = \langle T(u), v\rangle$. Hence we have the following

$$\begin{aligned} |\langle F, {}_{x_n}(v_i * \check{u}_i)\rangle| &= |\langle F, ({}_{x_n}v_i) * \check{u}_i\rangle| \qquad \text{(see Pier [21], pp. 14, (4))} \\ &= |\langle F(u_i), {}_{x_n}v_i\rangle| \\ &= \left|\int_G v_i(x_n t) F(u_i)(t)\,dt\right| \\ &\leq \int_G \|v_i\|_\infty {}_{x_n}1_V(t) |F(u_i)(t)|\,dt \\ &= \int_{x_n^{-1}V} \|v_i\|_\infty |F(u_i)(t)|\,dt \\ &\leq \|v_i\|_\infty \Big(\int_{x_n^{-1}V} 1^q\,dt\Big)^{\frac{1}{q}} \Big(\int_{x_n^{-1}V} |F(u_i)|^p\,dt\Big)^{\frac{1}{p}} \\ &= \|v_i\|_\infty \big(\lambda(V)\big)^{\frac{1}{q}} \Big(\int_{x_n^{-1}V} |F(u_i)|^p\,dt\Big)^{\frac{1}{p}} \longrightarrow 0 \end{aligned}$$

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as $n \longrightarrow \infty$ since $F(u_i) \in L^p(G)$ and for any compact subset K of G, $x_n^{-1}V$ is disjoint with K for sufficiently large n, where q is a real number so that $1/p + 1/q = 1$. ∎

REMARK 3.2. This lemma can also be derived from Lemma 8.4 of [17] as follows. Lemma 8.4 of [17] implies that we have not only the inclusion $A_p(G) \subseteq \mathrm{wap}\big(PF_p(G)\big)$ but also the inclusion $A_p(G) \subseteq \mathrm{wap}\big(PM_p(G)\big)$. It follows that, for $a \in A_p(G)$, the set $\{a_x : x \in G\}$ is relatively weakly compact ($a_x = \delta_x \circ a$; see our Proposition 4.4 below). The set $\{\delta_x : x \in G\}$, being the spectrum of the algebra $A_p(G)$, has the zero functional as the weak* cluster point if and only if G is not compact. Note that for any $f \in L^1(G)$ and $x \in G$, we have $a_x = \langle \delta_x \circ a, f\rangle = \big((\check{f}\check{\Delta})\big) * a(x)$, and $(\check{f}\check{\Delta}) * a \in A_p(G)$. It follows that if $\{x_n\}$ is any sequence in G that goes to "infinity", then $a_{x_n} \longrightarrow 0$ in the w^*-topology. So $a_{x_n} \longrightarrow 0$ weakly in $A_p(G)$ (hence ${}_{x_n}a = (\check{a}_{\check{x}_n^{-1}}) \longrightarrow 0$ weakly; see Theorem 4.3).

COROLLARY 3.3. *Let G be a locally compact group. If $A_p(G)$ has the Schur property then G is compact.*

PROOF. Suppose G is not compact. Let $a \in A_p(G)$ be any nonzero element and let $\{x_n\}$ be a sequence as in Lemma 3.1. Then ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$. But $\|{}_{x_n}a\| = \|a\| \not\rightarrow 0$. So $\{{}_{x_n}a\}$ does not converge in norm. Therefore $A_p(G)$ does not have the Schur property. ∎

COROLLARY 3.4. *Let G be a locally compact group. For any nonzero element $a \in A_p(G)$, $\{{}_xa : x \in G\}$ is a compact subset of $A_p(G)$ if and only if G is compact.*

PROOF. Let G be compact. Let $\{{}_{x_\alpha}a\}$ be any net in the orbit $\{{}_xa : x \in G\}$. Since G is compact, we can assume that $x_\alpha \longrightarrow x$ for some $x \in G$. Then ${}_{x_\alpha}a \longrightarrow {}_xa$ in $A_p(G)$ since translation is norm continuous (see Granirer and Leinert [12], pp. 466). Thus $\{{}_xa : x \in G\}$ is compact.

Conversely, suppose $\{{}_xa : x \in G\}$ is compact. If G is not compact, then we can choose a sequence $\{x_n\}$ from G as in Lemma 3.1. It follows from Lemma 3.1 that for any $a \in A_p(G)$, ${}_{x_n}a \longrightarrow 0$ weakly in $A_p(G)$. Since $\|{}_{x_n}a\| = \|a\|$, for any non-zero $a \in A_p(G)$, $\{{}_{x_n}a\}$ does not have any subsequence which is convergent in norm. This is a contradiction. Thus G is compact. ∎

REMARK 3.5. This may fail if $a \in B_p(G) \sim A_p(G)$. For example, if G is a noncompact locally compact group then $1 \in B_p(G)$. So $\{{}_x1 : x \in G\} = \{1\}$ is compact, although G is not compact.

COROLLARY 3.6. *Let G be an amenable locally compact group. If $B_p(G)$ has the RNP and the DPP then G is compact.*

PROOF. Since $B_p(G) = PF_p(G)^*$ and $B_p(G)$ has the RNP and the DPP, it follows from Theorem 3 of [7] that $B_p(G)$ has the Schur property. Hence $A_p(G)$, the subspace of $B_p(G)$, also has the Schur property. Corollary 3.3 implies that G is compact. ∎

REMARK 3.7. Lau and Ülger [17] showed that $B(G)$ has the RNP and the DPP if and only if G is compact.

For any $x \in G$ and $f \in PM_p(G)$, define ${}_xf \in PM_p(G)$ by $\langle {}_xf, u\rangle = \langle f, {}_{x^{-1}}u\rangle$ for all $u \in A_p(G)$. It follows clearly that $\|f\| = \|{}_xf\|$ since $\|{}_{x^{-1}}u\| = \|u\|$ for all $u \in A_p(G)$.

COROLLARY 3.8. *Let G be a locally compact group. If there is a sequence $\{x_n\}$ in G as in Lemma 3.1 then for any $f \in PM_p(G)$, ${}_{x_n}f \longrightarrow 0$ in the w^*-topology of $PM_p(G)$. Consequently, if there is a nonzero $f \in PM_p(G)$ such that the right multiplication operator $\gamma_f\colon PM_p(G) \longrightarrow PM_p(G)$ defined by $\gamma_f(g) = g \cdot f$ for $g \in PM_p(G)$ is compact then G is a compact group.*

PROOF. To prove the first claim, if $\{x_n\}$ in G is as in Lemma 3.1 then it follows from Lemma 3.1 that for any $u \in A_p(G)$, $\langle {}_{x_n}f, u\rangle = \langle f, {}_{x_n^{-1}}u\rangle \longrightarrow 0$.

For the second claim, suppose G is noncompact, and let γ_f be a compact operator as defined above. Note that for any $u \in A_p(G)$ and $x \in G$ we have $u\delta_x = {}_xu \in A_p(G)$, since for any $h \in L^1(G) \subseteq PF_p(G)$,

$$\langle u\delta_x, h\rangle = \langle u, \delta_x \cdot h\rangle = \langle u, {}_{x^{-1}}h\rangle = \langle {}_xu, h\rangle.$$

It follows that $\gamma_f(\delta_x) = {}_{x^{-1}}f$ since

$$\langle \gamma_f(\delta_x), u\rangle = \langle \delta_x \cdot f, u\rangle = \langle f, u\delta_x\rangle = \langle f, {}_xu\rangle = \langle {}_{x^{-1}}f, u\rangle.$$

Thus if ${}_{x_n^{-1}}f \longrightarrow 0$ in w^*-topology, then $\{\gamma_f(\delta_{x_n})\}$ has a subsequence $\{\gamma_f(\delta_{x_{nk}})\}$ which converges to 0 in norm by the compactness of the operator γ_f. This contradicts $\|{}_{x_{nk}^{-1}}f\| = \|f\| > 0$ for all k. ∎

That $A_p(G)$ has the RNP is close to $A_p(G)$ being a dual space (see Theorem 4.1 of Taylor [22]). Recall that for any $T \in PM_p(G)$, the support of T, denoted by $\operatorname{supp}(T)$, is a subset of G defined by $x \in \operatorname{supp}(T)$ if and only if for all $u \in A_p(G)$ we have $u \cdot T = 0$ implies $u(x) = 0$. Let E be a subset of G. Let $PM_p(G)|_E$ denote the subset of $PM_p(G)$ consisting all the elements $T \in PM_p(G)$ such that $\operatorname{supp}(T)$ is contained in E.

PROPOSITION 3.9. *Let G be a locally compact group and let G_0 be an open subgroup of G. If $A_p(G) = X^*$ for some Banach space X, then*

$$X \text{ is a subspace of } PM_p(G) \text{ and } A_p(G_0) = \big(X \cap PM_p(G_0)\big)^*.$$

PROOF. Observe that X is a subspace of $X^{**} = PM_p(G)$ and $A_p(G_0)$ can be identified with the subalgebra of $A_p(G)$ consisting of functions which vanish outside G_0 (see Herz [14], Proposition 5). Thus, $PM_p(G_0) \cong PM_p(G)|_{G_0}$ and every element in $A_p(G_0)$ is a bounded linear functional on $X \cap PM_p(G_0)$. Let $u \in \big(X \cap PM_p(G_0)\big)^*$. Then by Hahn-Banach Theorem, u can be extended to an element $\tilde{u}$ in X^*. It follows that $\tilde{u} \in X^* = A_p(G)$. Since for any $f \in X \cap PM_p(G_0)$, $\tilde{u}(f) = \tilde{u}|_{G_0}(f)$, we have $u = \tilde{u}|_{X \cap PM_p(G_0)} = \tilde{u}|_{G_0}$ and u is an element of $A_p(G_0)$ since $\tilde{u}|_{G_0} \in A_p(G_0)$ (see Herz [14], Proposition 5). ∎

COROLLARY 3.10. *Let G be a metrizable locally compact group. If $A_p(G)$ is a dual space then $A_p(G)$ has the RNP.*

PROOF. It suffices to show that any separable subspace of $A_p(G)$ has the RNP (see Huff [16], page 77). Since any separable subspace of $A_p(G)$ is contained in $A_p(G_0)$ for some open σ-compact subgroup of G, we only show that $A_p(G_0)$ has the RNP. If $A_p(G)$ is a dual space then it follows from Proposition 3.9 that $A_p(G_0)$ is a separable dual space. Hence $A_p(G_0)$ has the RNP. ∎

REMARK 3.11. Let G be a locally compact group. If $p = 2$ then $A_2(G)$ has the RNP implies that $A_2(G)$ is a dual space (see Taylor [22] Theorem 3.5).

4. Almost periodic functionals on $PF_p(G)$.

In this section we investigate when elements of $A_p(G)$ or $B_p(G)$ are almost periodic functionals on $PF_p(G)$. When $\ell^1(G) \subseteq PF_p(G)$, we will see that it is easy to prove our main result Theorem 4.7 by applying Lemma 3.1. We like to know exactly for which groups that $\ell^1(G) \subseteq PF_p(G)$ holds, that is, $PF_p(G)$ has an identity. The following is a consequence of results of Granirer [10].

PROPOSITION 4.1. *Let G be a locally compact group. $\ell^1(G) \subseteq PF_p(G)$ if and only if G is discrete.*

PROOF. It follows from the definition that if G is discrete then $\ell^1(G) = L^1(G) \subseteq PF_p(G)$.

Conversely, let $\ell^1(G) \subseteq PF_p(G)$. Then $\delta_e \in PF_p(G)$. Assume that G is separable first. So $PF_p(G)$ is separable. By the fact that δ_e is the identity of $PF_p(G)$ together with the Corollary of Granirer [10] on page 128, which says that $PF_p(G) \subset UC_p(\hat{G}) \subset C_p(\hat{G})$ for any locally compact group G, we have that $PF_p(G) = UC_p(\hat{G}) = C_p(\hat{G})$, where $UC_p(\hat{G}) = \overline{A_p(G) \cdot PM_p(G)}$ and $C_p(\hat{G}) = \{T \in PM_p(G) : Tf_1 + f_2T \in PF_p(G) \text{ if } f_1, f_2 \in PF_p(G)\}$. Theorem 16 of Granirer [10] implies that G is discrete if we take $PF_p(G)$ as X in that Theorem.

Let G be an arbitrary locally compact group with $\ell^1(G) \subseteq PF_p(G)$. Take any open-closed compactly generated subgroup G_0 of G. Then $\ell^1(G_0) \subseteq PF_p(G_0)$. In fact, $PM_p(G_0) \cong PM_p(G) \mid G_0$ (see Herz [14], Proposition 5). So $PF_p(G_0) \cong PF_p(G) \mid G_0$. For any $x \in G_0$, $\delta_x \in PF_p(G)$ implies $\delta_x \in PF_p(G_0)$. Therefore $\ell^1(G_0) \subseteq PF_p(G_0)$. If G_0 is discrete we are done. Otherwise there is a compact normal subgroup K in G_0 such that $\lambda(K) = 0$ and G_0/K is separable (see Theorem 8.7 of Hewitt and Ross [15]). Now we prove that $\ell^1(G_0/K) \subseteq PF_p(G_0/K)$. Let $x \in G_0$. So $\delta_x \in PF_p(G_0)$. Let $f_n \in L^1(G_0)$ and $\|\delta_x - f_n\| \longrightarrow 0$ in $PM_p(G_0)$. Let $A_p(G_0)_K$ be the subalgebra of $A_p(G_0)$ consisting of functions which are constant on cosets of K. Then $A_p(G_0/K) \cong A_p(G_0)_K$ by Proposition 6 of Herz [14]. If we consider δ_x and f_n as elements of $PM_p(G_0/K)$, $\|f_n - \delta_x\| \longrightarrow 0$ since $\|(f_n - \delta_x)|_{A_p(G_0)_K}\| \leq \|f_n - \delta_x\|$. So $\delta_{xK} \in PF_p(G_0/K)$ and $\ell^1(G_0/K) \subseteq PF_p(G_0/K)$. Thus, G_0/K is discrete. It follows that K is open. This is a contradiction since $\lambda(K) = 0$. ∎

REMARK 4.2. (a) If $A_p(G)^* = PF_p(G)$ then G is discrete since $\ell^1(G) \subseteq A_p(G)^*$ and so $\ell^1(G) \subseteq PF_p(G)$ always holds. This was proved by Forrest [9].

(b) Let G be a discrete group. Then $\ell^1(G) \subseteq PF_p(G)$. By Lemma 3.1, it is easy to prove that $\operatorname{ap}\big(PF_p(G)\big) = B_p(G)$ implies that G is compact. In fact, if G is not compact, let $\{x_n\}$ in G be a sequence as in Lemma 3.1 and $a \in A_p(G)$ be such that $a(e) = 1$. Then $\delta_{x_n} \circ a \longrightarrow 0$ weakly in $A_p(G)$ (see Proposition 4.4 below). But $\|\delta_{x_n} \circ a\| = \|a_{x_n}\| \geq a(e) \neq 0$. So $\{\delta_{x_n} \circ a\}$ does not have any subsequence converging in norm. Hence $a \notin \operatorname{ap}\big(PF_p(G)\big)$.

THEOREM 4.3. *Let G be a locally compact group. For any real numbers $p > 1$ and $q > 1$ with $1/p + 1/q = 1$, the operator $T: A_p(G) \longrightarrow A_q(G)$ defined by $T(u) = \check{u}$ is an isometric isomorphism of Banach algebras $A_p(G)$ onto $A_q(G)$. Hence $T^*: PM_q(G) \longrightarrow PM_p(G)$ is a linear isometry from $PM_q(G)$ onto $PM_p(G)$. Furthermore, the restriction T_R^* is a linear isometry from $PF_q(G)$ onto $PF_p(G)$.*

PROOF. By definition, $\|u\| = \inf \sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q$, where the infimum is taken over all possible representations $u = \sum_{n=1}^{\infty} v_n * \check{u}_n$ for $u_n \in L^p(G)$ and $v_n \in L^q(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$. Hence $T(u) = \check{u} = \sum_{n=1}^{\infty} u_n * \check{v}_n$ for $u_n \in L^p(G), v_n \in L^q(G)$ with $\sum_{n=1}^{\infty} \|u_n\|_p \|v_n\|_q < \infty$ (see Pier [21], pp. 14 (3)). So $T(u) \in A_q(G)$ and $\|T(u)\|_{A_q(G)} = \|u\|_{A_p(G)}$ by the definition. It is clear that T is an algebraic isomorphism and T^* is a linear isometry from $PM_q(G)$ onto $PM_p(G)$.

Now we show T_R^* is a linear isometry from $PF_q(G)$ onto $PF_p(G)$. Since $L^1(G)$ is dense in $PF_q(G)$ and T^* is a linear isometry of Banach algebras, it is sufficient to show that $T^*(f) \in PF_p(G)$ for all $f \in L^1(G) \subseteq PF_q(G)$. Let $f \in L^1(G)$ and $u \in A_p(G)$, then

$$\langle T^*(f), u\rangle = \int_G u(t^{-1}) f(t)\, dt = \int_G u(t) f(t^{-1}) \Delta(t^{-1})\, dt = \langle \check{f}\check{\Delta}, u\rangle.$$

So $T^*(f) = \check{f}\check{\Delta}$ is in $L^1(G) \subseteq PF_p(G)$. ∎

PROPOSITION 4.4. *Let G be a locally compact group. Then $\delta_x \circ u = u_x$ for all $x \in G$ and $u \in A_p(G)$. Furthermore, if $\{x_n\}$ is as in Lemma 3.1 and $a \in A_p(G)$ is any element, then $\delta_{x_n} \circ a \longrightarrow 0$ weakly in $A_p(G)$.*

PROOF. We show that $\delta_x \circ u = u_x$ first. For all $f \in L^1(G) \subseteq PF_p(G)$, by the definition,

$$\langle \delta_x \circ u, f\rangle = \langle u, f * \delta_x\rangle = \langle u, \Delta(x^{-1}) f_{x^{-1}}\rangle = \langle u_x, f\rangle.$$

Thus, $\delta_x \circ u = u_x$ since both $\delta_x \circ u$ and u_x are elements of $PF_p(G)^*$.

For any $F_p \in PM_p(G)$, by Theorem 4.3, there is an $F \in PM_q(G)$ such that $T^*(F) = F_p$. Hence, by applying Lemma 3.1 to $A_q(G)$, we have

$$\langle F_p, \delta_{x_n} \circ a\rangle = \langle T^*(F), a_{x_n}\rangle = \langle F, {}_{x_n^{-1}}\check{a}\rangle \longrightarrow 0. \qquad ∎$$

To prove our main theorem for general G, we need the following lemma which is a nondiscrete version of Lemma 3.1.

LEMMA 4.5. *Let G be a locally compact group. If there is a sequence $\{x_n\}$ in G as in Lemma 3.1 then $h_n \circ a \in A_p(G)$ and $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$, where $a \in A_p(G)$ is any element and $h_n = {}_{x_n}1_W \in PF_p(G)$ for any compact subset W of G with positive measure.*

PROOF. For all $f \in L^1(G) \subseteq PF_p(G)$, we have

$$\begin{aligned} \langle h_n \circ a, f\rangle &= \langle a, f * h_n\rangle \\ &= \int_G a(t) f(y)\, h_n(y^{-1}t)\, dy\, dt \\ &= \int_G a(t) f(y)\, \check{h}_n(t^{-1}y)\, dt\, dy \\ &= \langle a * \check{h}_n, f\rangle. \end{aligned}$$

Hence $h_n \circ a = a * \check{h}_n$. To show that $h_n \circ a \in A_p(G)$, let $a_k \in A_p(G)$ be with compact support and $\|a_k - a\| \longrightarrow 0$. Then $h_n \circ a_k \in A_p(G)$ and $\|h_n \circ a - h_n \circ a_k\| \leq \|h_n\|\,\|a_k - a\| \longrightarrow 0$ uniformly for n as $k \longrightarrow \infty$. Hence $h_n \circ a \in A_p(G)$. To prove that $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$, we can assume that the support of a is compact. So $a \in L^q(G)$, where q is a real number such that $1/p + 1/q = 1$. For any $F_p \in PM_p(G)$, by Theorem 4.3, there is an $F \in PM_q(G)$ such that $T^*(F) = F_p$. Thus we have

$$\begin{aligned}
|\langle F_p, h_n \circ a\rangle| &= |\langle T^*(F), a * \check{h}_n\rangle| \\
&= |\langle F, h_n * \check{a}\rangle| \qquad \text{(see Pier [21], pp. 14, (4))} \\
&= |\langle F(a), {}_{x_n}1_W\rangle| \\
&= \left|\int_{x_n^{-1}W} F(a)(t)\,dt\right| \\
&\leq \left(\int_{x_n^{-1}W} 1^p\,dt\right)^{\frac{1}{p}} \left(\int_{x_n^{-1}W} |F(a)|^q\,dt\right)^{\frac{1}{q}} \\
&= \big(\lambda(W)\big)^{\frac{1}{p}} \left(\int_{x_n^{-1}W} |F(a)|^q\,dt\right)^{\frac{1}{q}} \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$ since $F(a) \in L^q(G)$, and for any compact subset K of G, $x_n^{-1}W$ and K are disjoint for sufficiently large n. ■

REMARK 4.6. This lemma can also be derived from our Corollary 3.8 and Corollary 8.5 of Lau and Ülger in [17].

The following theorem improves some of the results of Lau and Ülger [17]. The equivalence (1) ⇔ (2) is a generalization of the equivalence *(a)* ⇔ *(j)* in Theorem 4.5 of Lau and Ülger [17]. It also improves Corollary 8.7 of Lau and Ülger [17]. Each of the equivalences (1) ⇔ (3) and (1) ⇔ (4) improves *(a)* ⇔ *(b)* in Theorem 8.8 of Lau and Ülger [17]. The equivalence (1) ⇔ (5) is in Theorem 8.8 of Lau and Ülger [17]; here we give a different proof for (5) ⇒ (1).

THEOREM 4.7. *Let G be a locally compact group. Then the following are equivalent*

(1) *G is compact.*
(2) $\mathrm{ap}\big(PF_p(G)\big) = B_p(G)$
(3) $A_p(G) \subseteq \mathrm{ap}\big(PF_p(G)\big)$
(4) $A_p(G) \cap \mathrm{ap}\big(PF_p(G)\big) \neq \{0\}$
(5) $A_p(G) \subseteq \mathrm{ap}\big(PM_p(G)\big)$.

PROOF. (1) ⇒ (2) In Corollary 8.7 of Lau and Ülger [17], it is proved that compactness of G implies $\mathrm{ap}\big(PF_p(G)\big) = B_p(G)$.

(2) ⇒ (3) ⇒ (4) It is clear since $A_p(G) \subseteq B_p(G)$ and $A_p(G) \neq \{0\}$.

(4) ⇒ (1) Let G be noncompact and let $a \in A_p(G) \cap \mathrm{ap}\big(PF_p(G)\big)$ be nonzero. Let W be a compact subset of G such that $1_W * \check{a}(e) = \int_G 1_W(t)\check{a}(t)\,dt \neq 0$ and let h_n be defined as in Lemma 4.5. Then $h_n \circ a \in A_p(G)$ and $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$. Since $a \in A_p(G) \cap \mathrm{ap}\big(PF_p(G)\big)$, we assume that $\|h_n \circ a\| \longrightarrow 0$ in the norm topology of $A_p(G)$ by passing to a subsequence. As $h_n \circ a = a * \check{h}_n$ (see the proof of Lemma 4.5) and $\delta_x \in A_p(G)^*$

has norm 1 for any $x \in G$, we have $\|h_n \circ a\| \geq \delta_{x_n}(a * \check{h}_n) = (h_n * \check{a})(x_n{}^{-1}) = 1_W * \check{a}(e) > 0$ for all n. This is a contradiction.

(1) ⇒ (5) It was proved in Theorem 8.8 of Lau and Ülger [17].

(5) ⇒ (1) If G is not compact, we choose $\{x_n\}$ from G as in Lemma 3.1 and $a \in A_p(G)$ with $a(e) > 0$. It follows from Proposition 4.4 that $\delta_{x_n} \circ a = a_{x_n}$ and $\delta_{x_n} \circ a \longrightarrow 0$ weakly in $A_p(G)$. Thus, $\delta_{x_n} \circ a \longrightarrow 0$ in the w^*-topology of $PM_p(G)^*$ as well. By (5), we assume that $\delta_{x_n} \circ a \longrightarrow 0$ in norm by passing to a subsequence. But $\|\delta_{x_n} \circ a\| = \|a_{x_n}\| \geq a(e) > 0$ for all n. This is impossible. Therefore, G is compact. ∎

REMARK 4.8. It is interesting to compare (1) ⇔ (3) and (1) ⇔ (4) of our Theorem 4.6 with Corollary 8.5 of Lau and Ülger [17], where it has been showed that $A_p(G) \subseteq \mathrm{wap}\big(PF_p(G)\big)$ for all locally compact groups G.

5. **Compactness of G and some operators.**

For any Banach algebra A, Arens [1], [2] introduced two natural Banach algebra products on A^{**}, each of which is an extension of the original product in A when A is canonically embedded in A^{**}. Duncan and Hosseiniun showed in Lemma 3 in [8] that A is an ideal under either Arens product if and only if all the multiplication operators on $PF_p(G)$ are weakly compact (see also Ülger [23] and Forrest [9]). Brian Forrest [9] proved that $A_p(G)$ is an ideal in its second dual if and only if G is discrete. The main result of this section is Theorem 5.2, which shows that $PF_p(G)$ is an ideal in its second dual if and only if G is compact. It also generalizes some results of Lau and Ülger [17]. For its proof we need the following lemma.

LEMMA 5.1. *Let G be a locally compact group. If $\{x_n\}$ is a sequence as in Lemma 3.1, then $\tau_f({}_{x_n}1_V) \longrightarrow 0$ and $\tau_f\big((1_V)_{x_n}\Delta(x_n)\big) \longrightarrow 0$ in the w^*-topology of $PM_p(G)$, where $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is defined by $\tau_f(h) = f \cdot h$ for all $h \in PF_p(G)$, V is any compact subset of G, and $f \in L^1(G)$.*

PROOF. Let $u \in A_p(G)$. By applying Corollary 3.8, we have

$$\langle \tau_f({}_{x_n}1_V), u\rangle = \langle {}_{x_n}1_V, (\check{\Delta}\check{f}) * u\rangle \longrightarrow 0 \text{ as } (\check{\Delta}\check{f}) * u \in A_p(G).$$

Note that $\langle \tau_f\big((1_V)_{x_n}\Delta(x_n)\big), u\rangle = \langle f * 1_V, u_{x_n^{-1}}\rangle$. By Theorem 4.3, let $f * 1_V = T_R^*(h)$ for some $h \in PF_q(G)$. Then $\langle \tau_f({}_{x_n}1_V), u\rangle = \langle T_R^*(h), u_{x_n^{-1}}\rangle = \langle h, {}_{x_n}\check{u}\rangle \longrightarrow 0$ by Lemma 3.1. ∎

In the following theorem, the equivalence (1) ⇔ (3) generalizes Theorem 6.7 (*a*) of Lau and Ülger [17], and the equivalence (1) ⇔ (5) is a parallel result of Forrest [9].

THEOREM 5.2. *Let G be a locally compact group. Then the following are equivalent:*

(1) G is compact.

(2) $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is compact for all $f \in PF_p(G)$, where $\tau_f(g) = f \cdot g$.

Furthermore, if G is amenable, then all the conditions from (1) to (5) are equivalent.

(3) $\tau_f\colon PF_p(G) \longrightarrow PF_p(G)$ is weakly compact for all $f \in PF_p(G)$, where $\tau_f(g) = f \cdot g$.

(4) $\gamma_f\colon PF_p(G) \longrightarrow PF_p(G)$ is weakly compact for all $f \in PF_p(G)$, where $\gamma_f(g) = g \cdot f$.

(5) $PF_p(G)$ is an ideal in its second dual under either Arens product.

PROOF. (1) $\Rightarrow$ (2) This follows from Lemma 8.6 of Lau and Ülger [17].

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) Let G be noncompact. Then we can have a sequence $\{x_n\}$ in G as in Lemma 5.1 (see Lemma 3.1). Let U and V be compact subsets of G with positive measures and $f = 1_U$.

Let (2) holds. As $\tau_f: PF_p(G) \longrightarrow PF_p(G)$ is compact and

$$\|(1_V)_{x_n}\Delta(x_n)\| \leq \|(1_V)_{x_n}\Delta(x_n)\|_1 = \lambda(V)$$

for all n, we may assume that $\tau_f\big((1_V)_{x_n}\Delta(x_n)\big) \longrightarrow 0$ in the norm topology of $PF_p(G)$ by passing to a subsequence since, by Lemma 5.1, it converges to 0 in the w^*-topology of $A_p(G)^*$. Hence we get

$$\begin{aligned}\big\|\tau_f\big((1_V)_{x_n}\Delta(x_n)\big)\big\| &= \big\|f * \big((1_V)_{x_n}\Delta(x_n)\big)\big\| \\ &= \|f * (1_V * \delta_{x_n^{-1}})\| \qquad \text{(see Pier [21], pp. 14(2))} \\ &\geq \|f * (1_V * \delta_{x_n^{-1}}) * \delta_{x_n}\| \\ &= \|f * 1_V\| > 0 \text{ for all } n.\end{aligned}$$

This is a contradiction.

Assume (3) is true. By a similar argument as above, we can assume that $\tau_f({}_{x_n}1_V) \longrightarrow 0$ in the weak topology of $PF_p(G)$. Then since $1 \in B_p(G) = PF_p(G)^*$,

$$\begin{aligned}\langle 1, \tau_f({}_{x_n}1_V)\rangle &= \int_G f(t)({}_{x_n}1_V)(t^{-1}y)\,dt\,dy \\ &= \int_G 1_U(t)({}_{x_n}1_V)(t^{-1}y)\,dy\,dt \\ &= \lambda(U)\lambda(V) > 0.\end{aligned}$$

This is a contradiction.

(1) $\Rightarrow$ (3) is trivial since (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4) follows from Lemma 8.6 of Lau and Ülger [17].

(4) $\Rightarrow$ (1) Let $T_R^*: PF_p(G) \longrightarrow PF_q(G)$ be the linear isometry as in Theorem 4.3. We prove that $T_R^*(f) \cdot g = T_R^*\big((\check{\Delta}\check{g}) \cdot f\big)$ for any $f \in PF_p(G)$ and $g \in PF_q(G)$. Assume that $f, g \in L^1(G)$ and $u \in A_q(G)$. Then, since $g \circ u = u * \check{g}$ (see the proof of Lemma 4.5),

$$\begin{aligned}\langle T_R^*(f) \cdot g, u\rangle &= \langle T_R^*(f), g \circ u\rangle \\ &= \langle f, T_R(g \circ u)\rangle \\ &= \langle f, T_R(u * \check{g})\rangle \\ &= \langle f, g * \check{u}\rangle\end{aligned}$$

$$\begin{aligned}\langle T_R^*\big((\check{\Delta}\check{g}) \cdot f\big), u\rangle &= \langle (\check{\Delta}\check{g}) * f, \check{u}\rangle \\ &= \langle f, g * \check{u}\rangle.\end{aligned}$$ ■

That $\{g \cdot f : \|g\| \leq 1, g \in PF_p(G)\}$ is weakly compact implies that $\{T_R^*(f) \cdot g : \|g\| \leq 1, g \in PF_q(G)\}$ is weakly compact. So G is compact by (3) $\Leftrightarrow$ (1).

(1) $\Leftrightarrow$ (5) follows from (3), (4) and Lemma 3 of Duncan and Hosseiniun [8] (p. 318). ■

COROLLARY 5.3. *Let G be a locally compact group. If every bounded linear operator $T\colon PF_p(G) \longrightarrow B_p(G)$ is compact then G is compact. Conversely, if G is compact then every bounded linear operator $T\colon PF_p(G) \longrightarrow B_p(G)$ is weakly compact.*

PROOF. Suppose that every bounded linear operator $T\colon PF_p(G) \longrightarrow B_p(G)$ is compact and G is not compact. Let $h_n \in PF_p(G)$ and $a \in A_p(G)$ be as in Lemma 4.5. Define $T\colon PF_p(G) \longrightarrow B_p(G)$ by $T(f) = f \circ a$ for all $f \in PF_p(G)$. Then T is a bounded linear operator. So it is a compact operator. Thus, we assume $h_n \circ a \longrightarrow u$ in norm for some $u \in B_p(G)$ by passing to a subsequence. By Lemma 4.5, $h_n \circ a \longrightarrow 0$ weakly in $A_p(G)$. Since $\|u\|_{B_p(G)} \le \|u\|_{A_p(G)}$ if $u \in A_p(G)$, $h_n \circ a \longrightarrow 0$ weakly in $B_p(G)$ as well. So $h_n \circ a \longrightarrow 0$ in the norm of $B_p(G)$. But $\|h_n \circ a\| \ge \delta_{x_n}(a * \check{h}_n) = (h_n * \check{a})(x_n{}^{-1}) = 1_W * \check{a}(e) > 0$ for all n if we take W and a properly. This is a contradiction.

Conversely, let G be compact. Suppose $T\colon PF_p(G) \longrightarrow B_p(G)$ is any bounded linear operator. To show that T is weakly compact, let $f_n \in PF_p(G)$ with $\|f_n\| \le 1$. Since G is compact, $B_p(G)$ has the RNP by Theorem 2 of Granirer [11]. So the predual $PF_p(G)$ has no copy of ℓ^1. By applying Rosenthal's ℓ^1-theorem, we can assume that $\{f_n\}$ is a weak Cauchy sequence by passing to a subsequence. Next, we show that $\{T(f_n)\}$ is a weak Cauchy sequence. Let $\alpha \in B_p(G)^*$. Define $\tilde{\alpha} \in PF_p(G)^*$ by $\tilde{\alpha}(f) = \langle \alpha, T(f) \rangle$. Then $\tilde{\alpha}$ is bounded since both T and α are bounded. So $\{\tilde{\alpha}(f_n)\}$ is a Cauchy sequence, *i.e.* $\{\langle \alpha, T(f_n)\rangle\}$ is a Cauchy sequence. Since G is compact, $A_p(G) = B_p(G)$ is weakly sequentially complete by Lemma 18 of Granirer [10]. Thus there is an $b \in B_p(G)$ such that $T(f_n) \longrightarrow b$ weakly in $B_p(G)$. ■

COROLLARY 5.4. *Let G be a discrete group. Then every bounded linear operator $T\colon PF_p(G) \longrightarrow B_p(G)$ is compact if and only if G is finite.*

REMARK 5.5. For discrete groups, this generalizes the equivalence $(a) \Leftrightarrow (h)$ in Theorem 4.5 of Lau and Ülger [17], where this result for $p = 2$ was proved by using C^*-algebraic techniques.

REFERENCES

1. R. Arens, *Operations induced in function classes*, Monatsh. Math. **55**(1951), 1–19.
2. ———, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2**(1951), 839–848.
3. R. Bourgin, *Geometric aspectsof convex sets with the Radon-Nikodym property*, Lecture Notes in Math. **993**, Springer-Verlag, 1983.
4. L. J. Bunce, *The Dunford-Pettis property in the predual of a Von Neumann algebra*, Proc. Amer. Math. Soc. **116**(1992), 99–100.
5. M. Cowling, *An application of Littlewood-Paley theory in harmonic analysis*, Math. Ann. **241**(1979), 83–96.
6. J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Math., Springer-Verlag, New York, 1984.
7. ———, *A survey of results related to the Dunford-Pettis property*, Contemporary Math. **2**(1980), 15–60.
8. J. Duncan and S. A. R. Hosseiniun, *The second dual of Banach algebra*, Proc. Roy. Soc. Edinburgh **84A**(1979), 309–325.
9. B. Forrest, *Arens regularity and discrete groups*, Pacific J. Math. **151**(1991), 217–227.
10. E. E. Granirer, *On some spaces of linear functionals on the algebras $A_p(G)$ for locally compact groups*, Colloq. Math. **52**(1987), 119–132.

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11. ———, *An application of the Radon-Nikodym property in harmonic analysis*, Bull. Un. Mat. Ital. B(5)**18** (1981), 663–671.

12. E. E. Granirer and M. Leinert, *On some topologies which coincide on the unit sphere of the Fourier-Stieltjes algebra B(G) and of the measure algebra M(G)*, Rocky Mountain J. Math. **11**(1981), 459–472.

13. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, New York, 1969.

14. C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23**(1973), 91–123.

15. E. Hewitt and K. Ross, *Abstract harmonic analysis, Vol. I*, Springer-Verlag, Berlin, 1963.

16. R. Huff, *The Radon-Nikodym property*, Contemporary Math. **2**(1980), 75–89.

17. A. T. Lau and A. Ülger, *Some geometric properties on the Fourier and Fourier Stieltjes algebras of locally compact groups, Arens regularity and related problems*, Trans. Amer. Math. Soc. **337**(1993), 321–359.

18. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, New York, 1984.

19. T. W. Palmer, *Classes of nonabelian, noncompact locally compact groups*, Rocky Mountain J. Math. **8**(1973), 683–741.

20. A. L. T. Paterson, *Amenability*, Amer. Math. Soc., Providence, Rhode Island, 1988.

21. J. P. Pier, *Amenable Locally Compact Groups*, Wiley, New York, 1984.

22. K. Taylor, *Geometry of Fourier algebras and locally compact groups with atomic representations*, Math. Ann. **262**(1983), 183–190.

23. A. Ülger, *Arens regularity sometimes implies the RNP*, Pacific J. Math. **143**(1990), 377–399.

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SUFFICIENTLY HOMOGENEOUS CLOSED EMBEDDINGS OF $\mathbb{A}^{n-1}$ INTO $\mathbb{A}^n$ ARE LINEAR

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ABSTRACT. We show that over a field k of characteristic zero an affine $(n-1)$-space $\mathbb{A}_k^{n-1}$ embedded as a closed subvariety in affine n-space $\mathbb{A}_k^n$ and homogeneous for a codimension two linear torus action on $\mathbb{A}_k^n$ is defined by the vanishing of a variable.

To determine the nature of closed embeddings of affine m-space $\mathbb{A}^m$ into affine n-space $\mathbb{A}^n$ is a central problem in the geometry of affine spaces. The famous "epimorphism theorem" of Abhyankar-Moh [AM] and Suzuki [S] settles the case $m = 1, n = 2$ over a field k of characteristic zero: A "line" $\mathbb{A}_k^1$ embedded in the "plane" $\mathbb{A}_k^2$ is defined by the vanishing of a variable. (A *variable* on $\mathbb{A}^n$ is a function $y = y_1$ that extends to a global coordinate system $y_1, y_2, \ldots, y_n$ for $\mathbb{A}^n$.) In most other cases, only very partial results have been obtained so far, even when, or better particularly when $m = n - 1$ (see [RS], for instance). With the epimorphism theorem as a starting point we will show that if $\operatorname{char} k = 0$ and if $Y \simeq \mathbb{A}_k^{n-1}$ is closed in $X \simeq \mathbb{A}_k^n$ and *sufficiently homogeneous* in the sense that it is invariant under a $(n-2)$-torus $T \simeq \mathbb{G}_{m,k}^{n-2}$ acting linearly and effectively on X, then again Y is defined by a variable. (Special cases of this result with $n = 3$ were considered in [PTD]). Our proof, an induction taking off from $n = 2$ and the epimorphism theorem, quite naturally forces us to consider certain finite group actions in addition to those of tori. We show:

MAIN THEOREM. *Let k be a field of characteristic zero and suppose*

$$\mathbb{A}_k^{n-1} \simeq Y \subset X \simeq \mathbb{A}_k^n$$

with Y closed in X and invariant under an effective linear action of

$$G = T \times F$$

where $T \simeq \mathbb{G}_{m,k}^r$ is a torus of dimension $r \geq n-2$ and F a finite abelian group. Then there exists a system of G-semi-invariant variables $y_1, \ldots, y_n$ for X such that Y is the zero locus of y_1.

In a sense this is only a modest contribution to the embedding problem since it begs the main question for our homogeneous situation, namely: "*Is every effective action of a $(n-2)$-torus on affine n-space linearizable*?" We were motivated to undertake the present

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study largely to gain some understanding of the difficulties one is likely to encounter trying to resolve this question by an inductive procedure along the lines of [KR1], starting with the expected positive answer for $n = 3$ (see [KR3], [KR4]).

1. **Notation and preliminary results.** We fix a field k of characteristic zero and set

$$A = k^{[n]} \text{ (the polynomial ring in } n \text{ variables over } k),$$
$$X = \mathbb{A}^n_k = \operatorname{Spec} A,$$
$$T = \mathbb{G}^r_{m,k} \text{ the } r\text{-dimensional torus over } k).$$

Let F be a finite abelian group and suppose

$$G \simeq T \times F$$

acts linearly and effectively on X. We call $n - r$ the codimension of the action. Let

$$\hat{G} \simeq \hat{T} \times \hat{F}$$

be the character group of G. We note that T is canonically determined as a subgroup of G and $\hat{T}$ as a quotient of $\hat{G}$ and that $r = \operatorname{rank} \hat{G}$. The action of G can be diagonalized, that is

$$A = k[x_1, \ldots, x_n] \tag{1.1}$$

such that

$$t \cdot x_i = \chi_i(t) x_i$$

with $\chi_i \in \hat{G}$. Moreover, $\chi_1, \ldots, \chi_n$ generate $\hat{G}$ (since the action is effective).

Hence, with respect to a fixed diagonalization, the action of G is given by a surjective homomorphism

$$\begin{gathered} \varphi : \mathbb{Z}^n \longrightarrow \hat{G} \\ \alpha = (a_1, \ldots, a_n) \longmapsto \chi^\alpha = \chi_1^{a_1} \ldots \chi_n^{a_n} \end{gathered} \tag{1.2}$$

and the action of G is in fact determined by

$$E = \operatorname{Ker} \varphi \subset \mathbb{Z}^n.$$

1.3. Definition - Lemma. Let

$$f = \sum f_\alpha x^\alpha \in A = k[x_1, \ldots, x_n].$$

(We use the shorthand notation $\alpha = (a_1, \ldots, a_n)$ and $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$). Then f is a semi-invariant for G if and only if

$$\operatorname{Supp} f = \{\alpha \mid f_\alpha \neq 0\}$$

is contained in a coset

$$\alpha^* + E$$

of E. We call $\chi = \varphi(\alpha^*)$ the weight of f.

1.4. Suppose χ_n is of infinite order in $\hat{G}$. The stabilizer G' of $(0, \ldots, 0, 1)$ then acts on $X' = x_n^{-1}(1) \simeq \mathbb{A}_k^{n-1}$, and there is an essentially bijective correspondence between G-homogeneous functions on X and G'-homogeneous functions on X' which we now describe.

Let $p\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-1}$, $p(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1})$ be the projection on the first $n-1$ factors. We obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, "\rho"'] & \mathbb{Z}^n \arrow[r, "\varphi"] \arrow[d, "p"'] & \hat{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E' \arrow[r] & \mathbb{Z}^{n-1} \arrow[r, "\varphi'"] & \hat{G}/\langle \chi_n \rangle \arrow[r] & 0
\end{tikzcd}$$

where, by definition,

$$\varphi'(a_1, \ldots, a_{n-1}) = \chi_1^{a_1} \cdots \chi_{n-1}^{a_{n-1}} \pmod{\chi_n}$$

and

$$E' = \operatorname{Ker} \varphi'.$$

Since χ_n has infinite order, a_n is uniquely determined by $(a_1, \ldots, a_{n-1})$ for any $(a_1, \ldots, a_n) \in E$. Hence

$$\rho\colon E \longrightarrow E'$$

is an isomorphism and there is a unique homomorphism

$$u\colon E' \longrightarrow \mathbb{Z}$$

such that

$$\rho^{-1}(a_1, \ldots, a_{n-1}) = \big(a_1, \ldots, a_{n-1}, -u(a_1, \ldots, a_{n-1})\big)$$

for $(a_1, \ldots, a_{n-1}) \in E'$.

$\hat{G}/\langle \chi_n \rangle$ has rank $r-1$ and hence is the character group of $G' = T' \times F'$, where T' is a $(r-1)$-torus and F' is finite abelian. It is clear that G' is the stabilizer of $(0, \ldots, 0, 1) \in X$ and that φ', or E', describes the action of G' on $X' = x_n^{-1}(1)$ in the sense of 1.2. Moreover, if

$$H(x_1, \ldots, x_{n-1}, x_n) \in k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}]$$

is G-homogeneous of weight $\chi \in \hat{G}$, then $H(x_1, \ldots, x_{n-1}, 1)$ is G'-homogeneous of weight $\bar{\chi} = \chi (\operatorname{mod} \chi_n) \in \hat{G}'$. Conversely, let

$$h(x_1, \ldots, x_{n-1}) = \sum h_\beta x^\beta \in k[x_1, \ldots, x_{n-1}]$$

be G'-homogeneous of weight $\bar{\chi} \in \hat{G}'$. Pick $\alpha^* = (a_1^*, \ldots, a_{n-1}^*)$ such that $\operatorname{Supp} h \subset \alpha^* + E'$. Then

$$H(x_1, \ldots, x_{n-1}, x_n) = \sum h_\beta x^\beta x_n^{-u(\beta-\alpha^*)} \in k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}]$$

is G-homogeneous of weight $\varphi(a_1^*, \ldots, a_{n-1}^*, 0)$.

1.5. REMARK. (i) We have on $X' \times G$

$$H(x_1, \ldots, x_{n-1}, \chi_n) = \chi_1^{-a_1^*} \cdots \chi_{n-1}^{-a_{n-1}^*} h(\chi_1 x_1, \ldots, \chi_{n-1} x_{n-1})$$

and H is uniquely determined by this relation.

(ii) E is a submodule of finite index in $\tilde{E}$, the kernel of $\mathbb{Z}^n \longrightarrow \hat{G} \longrightarrow \hat{T} \simeq \mathbb{Z}^r$. Let γ_i be the image of χ_i in $\mathbb{Z}^r$. Then for $\beta = (b_1, \ldots, b_{n-1}) \in \operatorname{Supp} h$ the unique corresponding term in H is

$$h_\beta x^\beta x_n^{a_n} \text{ with } - a_n \gamma_n = (b_1 - a_1^*)\gamma_1 + \cdots + (b_{n-1} - a_{n-1}^*)\gamma_{n-1}.$$

(iii) We have established a bijective correspondence between G-homogeneous elements H of $k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}]$, up to multiplication by a power of x_n, on the one hand and G'-homogeneous elements h of $k[x_1, \ldots, x_{n-1}]$ on the other.

(iv) In the passage from h to H there is a (generally non unique) choice of α^* that will lead to $H \in k[x_1, \ldots, x_{n-1}, x_n]$ and not divisible by x_n, and the resulting H is unique.

Let the notation be as in 1.1 and 1.5. In general, for $\beta_1, \ldots, \beta_s \in \mathbb{Z}^r$, we denote by $\langle \beta_1, \ldots, \beta_s \rangle_+$ the subsemigroup they generate. We write Γ for the semigroup of the action of T on X and note ([KbR])

$$\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle_+.$$

We recall that the following two conditions are equivalent ([KbR], [BH]).

1.6. (i) Γ is *unmixed*, that is

$$0 \neq \gamma \in \Gamma \Longrightarrow -\gamma \notin \Gamma,$$

(ii) the action of T on X is *fixpointed*, that is, every T-orbit has a fixpoint in its closure.

1.7. DEFINITION - LEMMA. Let the notation be as in 1.1 and 1.5. We denote by M the set of nonconstant T-invariant monomials in $x_1, \ldots, x_n$ and let

$$\pi: X \longrightarrow X//T = \operatorname{Spec} A^T$$

be the canonical map.

(i) $A^T = k[M]$.

(ii) The *nullcone* (at 0) is defined to be

$$X_0 = \pi^{-1}\big(\pi(0)\big).$$

X_0 is the union of all T-orbits with 0 in their closure, and the ideal of X_0 in A is generated by M. Any irreducible component of X_0 is a linear subspace of X with unmixed T-action.

(iii) An irreducible $x \in A$ is called a *nullvariable* if $x^{-1}(0) \subset X_0$ or, equivalently, if x divides every element of M. In particular, x is a variable and in fact is part of any T-homogeneous system of variables for X. Moreover, a nullvariable is G-homogeneous.

(iv) Let $Y \subset X$ be closed and T-invariant. Then $Y//T$ is naturally identified with a closed subset of $X//T$ and $Y_0 = X_0 \cap Y$.

PROOF. Only the G-homogeneity of a nullvariable needs commenting upon. It follows from: (a) The finite group F permutes the codimension 1 components of X_0. (b) Two nonproportional nullvariables have different T-weights. If not, one could replace the first by the second wherever it occurs in an invariant monomial and thereby construct an invariant monomial not divisible by the first.

1.8. REMARK. Suppose T acts with codimension ≤ 2 and let $Y \simeq \mathbb{A}_k^{n-1}$ be closed in X with $Y \not\subset x_i^{-1}(0), i = 1, \dots, n$. Then T acts effectively on Y with codimension ≤ 1, and hence this action is linearizable by [BB]. In particular, the action has a fixpoint, which we may assume to be the origin of X.

Our next two statements are certainly well known to the specialists. We include them for lack of proper references.

1.9. PROPOSITION ("EQUIVARIANT EPIMORPHISM THEOREM"). *Let the group G act linearly on $k[x,y]$ and suppose $f \in k[x,y]$ is G-semi-invariant and defines a line (i.e. $k[x,y]/f \simeq k^{[1]}$). Then there exists a G-semi-invariant $g \in k[x,y]$ such that $k[x,y] = k[f,g]$.*

PROOF. The proof of the epimorphism theorem given in [A] shows: We may assume f is monic in y and $\deg_y f = \deg f$. Then there is a divisor d of $\deg_y f$ such that $k[x,y] = k[f,g]$, where g is the d-approximate root *w.r.t.* y of f. By the uniqueness properties of approximate roots, g is G-semi-invariant.

1.10. LEMMA. *Let B be a factorial affine k-domain and K its field of quotients. Let A be a B-algebra and $u_1, \dots, u_r \in A$ such that*

$$K \otimes_B A = K[u_1, \dots, u_r].$$

Suppose that $u_1, \dots, u_r$ are algebraically independent mod mA for every maximal ideal m of B. Then

$$A = B[u_1, \dots, u_r].$$

PROOF. For each m,

$$(B/m)[u_1, \dots, u_r] \longrightarrow A/mA$$

is injective by assumption, that is, $mA \cap B[u_1, \dots, u_r] = mB[u_1, \dots, u_r]$. Given $x \in A$, there exists $p \in B$ such that $px \in B[u_1, \dots, u_r]$. Hence if $m \ni p$, $px \in B[u_1, \dots, u_r] \cap pA \subset B[u_1, \dots, u_r] \cap mA = mB[u_1, \dots, u_r]$. It follows that $px \in \bigcap_{m \ni p} mB[u_1, \dots, u_r] = \tilde{p}B[u_1, \dots, u_r]$, where $\tilde{p}$ is the product of the distinct prime divisors of p. If we choose p with a minimal number of prime divisors, we have $p = 1$.

2. **The cases $\dim X//T = 0$ and $r + \dim X^T \geq n-1$.** We give part of the proof of the main theorem in this section. We note that

$$\dim X^T \leq \dim X//T \leq n-r \leq 2.$$

We keep the notation of 1.1 and 1.7. We will assume tacitly that $0 \in Y$ and that none of $x_1, \dots, x_n$ vanishes identically on Y.

2.1. If $\dim X//T = 0$, then the action of T on X is unmixed and no T-weight γ_i is 0. Hence if x_1, say, appears as a monomial in the equation H of Y, we have, up to multiplication by a constant, $H = x_1 + \tilde{H}$ with $\tilde{H} \in k[x_2, \dots, x_n]$ and G-homogeneous.

2.2. LEMMA. *(i) Let $Y \subset X$ be an irreducible, G-stable closed hypersurface such that $Y^T = X^T$ and Y is smooth along X^T. If*

$$r + \dim X^T \geq n-1,$$

then the defining equation for Y is part of a G-homogeneous system of variables for X.

(ii) Suppose $Y \simeq \mathbb{A}^{n-1}, r = n-2$ and $\dim Y^T \geq 1$. Then the defining equation for Y is part of a G- homogeneous system of variables for X.

PROOF. In [KR1], Lemma 1.1 and Lemma 1.2, T-homogeneous variables defining Y are constructed, and it remains to verify only that they are part of a G-homogeneous system of variables. As to (i), we observe that if $f = \alpha x_{r+1} - \beta x_r$ is the equation for Y as in the proof of [KR1], Lemma 1.1, then α and β are G-homogeneous (and coprime), and we can choose a and b G-homogenous such that $a\alpha + b\beta = 1$. Then $g = bx_{r+1} + ax_r$ is G-homogeneous as well. As to (ii), we note that in the case (i) of the proof of [KR1], Lemma 1.2, $\mathbb{A}^1 \simeq Y^T \subset X^T \simeq \mathbb{A}^2$ is G-stable for the natural action of G on X^T and hence, by 1.9, defined by a variable that is part of a G-homogeneous system of variables for X^T.

2.3. By 2.2, the main theorem is proven in the following cases:

(i) $r \geq n-1$,

(ii) $\dim X^T = 2$,

(iii) $\dim X^T = \dim Y^T = 1$.

2.4. Proof of the main theorem in case $\dim X^T = 1$ and $\dim Y^T = 0$. There is a unique variable, x_1 say, with T-weight 0, and we have a G-equivariant decomposition $X = X^T \times V$ with $X^T = \operatorname{Spec} k[x_1]$, $V = \operatorname{Spec} k[x_2, \dots, x_n]$.

(α) Suppose $\dim X//T = 1$. Then $X^T = X//T$ and $Y = 0 \times V$.

(β) Suppose $\dim X//T = 2$. Then $X//T = X^T \times V//T$ with $V//T = \operatorname{Spec} k[m]$, where $m = x_2^{a_2} \cdots x_n^{a_n}$ is T-invariant. Now $\dim Y//T \leq 1$ since T acts effectively with codimension 1 on Y, and $\dim Y//T = 0$ is not possible since otherwise $Y \subset X_0$, which is of dimension $\leq n-2$. Hence $Y//T \simeq \mathbb{A}^1$. Moreover, since Y is smooth and Y^T is connected, $Y//T$ and X^T meet normally in one point in $X//T = \operatorname{Spec} k[x_1, m]$, with $m = 0$ as equation for X^T. Hence the equation for $Y//T$ in $X//T$ is of the form $x_1 + \phi(m)$ with $\phi(m) \in k[m]$, and the equation for Y in X is $x_1 + \phi(x_2^{a_2} \cdots x_n^{a_n})$, so clearly part of G-homogeneous system of variables for X.

3. **The main induction.** We continue the proof of the main theorem. By the results of Section 2 it remains to consider the case

$$\dim X//T > 0 \text{ and } \dim X^T = 0.$$

Note that now $X^T = \{0\}$ and $0 \in Y$. We keep in force the assumption $Y \not\subset x_i^{-1}(0)$, $i = 1,\dots,n$.

3.1. LEMMA. *Suppose x_n is a nullvariable on X and*

$$H = x_n + \tilde{H}$$

is irreducible and G-homogeneous with $\tilde{H}(0) = 0$ and $(0,\dots,0,1) \notin \operatorname{Supp}\tilde{H}$. Then $\tilde{H} = 0$.

PROOF. Since x_n is a nullvariable, there exists

$$(*) \qquad \alpha = (a_1,\dots,a_n) \in E, \quad a_1,\dots,a_{n-1} \geq 0, \quad a_n > 0.$$

On the other hand, if $\tilde{H} \neq 0$, $\tilde{H}$ has a monomial not divisible by x_n. This gives a relation $\chi_n = \chi_1^{c_1} \cdots \chi_{n-1}^{c_{n-1}}$ and hence an element

$$(**) \qquad \gamma = (c_1,\dots,c_{n-1},-1) \in E, \quad c_1,\dots,c_{n-1} \geq 0.$$

Since x_n is a nullvariable, the action of T on $\{x_n = 0\}$ is unmixed (see 1.7 (ii)), that is, the semi group $\Gamma' = \langle \gamma_1,\dots,\gamma_{n-1}\rangle_+$ is unmixed. But $(*)$ and $(**)$ give $-a_n\gamma_n \in \Gamma'$ and $a_n\gamma_n \in \Gamma'$, and this is not possible. So $\tilde{H} = 0$.

3.2. LEMMA. *If $x \in A$ is a nullvariable for X, then $y = x|Y$ is a nullvariable for Y.*

PROOF. By 1.7(iii), x is one of the x_i. Say $x = x_n$. By assumption, $Y \not\subset X_n = x_n^{-1}(0)$ and hence $Z = X_n \cap Y$ consists of codimension one components of the nullcone Y_0 of Y, all passing through the origin (see 1.7(iv)). If L is the linear part of the equation H of Y, then L is linearly independent of x_n by 3.1, that is, Y and X_n meet normally in the origin in one irreducible component. It follows that y is irreducible on Y and hence a nullvariable by 1.7(iii).

3.3. LEMMA. *There exists $x \in A$, x a nullvariable for X.*

PROOF. (i) Suppose $\dim X//T = 1$. Then $X//T = k[m]$, where m is an invariant monomial, and any x_i dividing m is a nullvariable on X.

(ii) Suppose $\dim X//T = 2$. Then $E \subset \mathbb{Z}^n$, the module of relations among the G-characters of the action (see 1.2) is of rank 2 and contains two linearly independent elements $\alpha = (a_1,\dots,a_n)$ and $\beta = (b_1,\dots,b_n)$ with non-negative coefficients, *i.e.*

$$\alpha,\beta \in E_+ = E \cap (\mathbb{Z}_+)^n.$$

For $\gamma = (c_1,\dots,c_n) \in \mathbb{Z}^n$, put $\operatorname{supp}\gamma = \{i \mid c_i \neq 0\}$. Choose $j \in \operatorname{supp}\beta$ such that $a_j/b_j = \min_i\{a_i/b_i\}$. Then $\alpha' = b_j\alpha - a_j\beta \in E_+$, α' and β are linearly independent, and $j \notin \operatorname{supp}\alpha'$. So we may assume to begin with that $\operatorname{supp}\beta \not\subset \operatorname{supp}\alpha$ and, repeating the argument if necessary, that $\operatorname{supp}\alpha \not\subset \operatorname{supp}\beta$ as well. Then every element of E_+ is a linear combination with non-negative rational coefficients of α and β and hence x_i is a nullvariable for X if and only if $i \in \operatorname{supp}\alpha \cap \operatorname{supp}\beta$.

So assume $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. We may assume that the equation of Y is of the form

$$H = x_n + \tilde{H}$$

where $\tilde{H} \neq 0$ has a monomial not divisible by x_n. As in the Proof of 3.1 we obtain an element

$$\gamma = (c_1, \ldots, c_{n-1}, -1) \in E$$

with $c_i \geq 0$, $i = 1, \ldots, n-1$. We claim that α, β, γ are linearly independent, in contradiction to $\operatorname{rank} E = 2$. In fact, this is obvious if $n \notin \operatorname{supp}\alpha \cup \operatorname{supp}\beta$. So suppose $n \in \operatorname{supp}\alpha$, say. Now $\operatorname{supp}\alpha$ has at least two elements (since $X^T = \{0\}$) and we may assume $n-1 \in \operatorname{supp}\alpha$ as well, that is, $\alpha = (a_1, \ldots, a_{n-1}, a_n)$ with $a_{n-1} > 0$, $a_n > 0$. A nontrivial relation $a\alpha + b\beta + c\gamma = 0$ is then not compatible with $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$.

3.1. *The induction.* Suppose now $n \geq 3$. In view of 3.2 and 3.3 we may assume that x_n, say, is a nullvariable on X and Y and that the linear part of the equation H for Y does not depend on x_n. Put $X' = x_n^{-1}(1)$ and $Y' = X' \cap Y$. We are then in the situation of 1.4. By 1.8, the action of G on Y is linearizable and by 1.7(iii), x_n restricts to a variable on Y. Hence $Y' \simeq \mathbb{A}^{n-2}$. By induction on n (for $n = 2$ we invoke the equivariant epimorphism theorem) we may assume that

$$h(x_1, \ldots, x_{n-1}) = H(x_1, \ldots, x_{n-1}, 1)$$

is part of a G'-homogeneous system of variables

$$h_1 = h, h_2, \ldots, h_{n-1}$$

for X'. Say x_1 appears in the linear part of H. Free to modify x_1 by x_1-free terms in H, we may assume to begin with that no term in H is of the form $x_i x_n^c$, $c \geq 0$, $i \geq 2$. Also, a term $x_1 x_n^c$, $c \geq 1$, cannot appear since otherwise $\gamma_n = 0$. Hence $h_1 = x_1 + \tilde{h}_1$ with $\operatorname{ord}_0 \tilde{h}_1 > 1$. After a G'-homogeneous linear change of variables we may then assume that $h_i = x_i + \tilde{h}_i$ with $\operatorname{ord}_0 \tilde{h}_i > 1$ for $i = 2, \ldots, n-1$ as well.

Now pick $\alpha_i^* = (0, \ldots, 1, \ldots, 0)$ (a 1 in place i) and let H_i be defined by h_i and α_i^* as in 1.4. Then $H_1 = H$ (see 1.5(iv)). Let $i \geq 2$ and consider $\beta = (b_1, \ldots, b_{n-1}) \in \operatorname{Supp} h_i$. By 1.5(ii), the corresponding monomial in H_i is $x^\beta x_n^{b_n}$ with

$$(*) \qquad -b_n\gamma_n = b_1\gamma_1 + \cdots + (b_i - 1)\gamma_i + \cdots + b_{n-1}\gamma_{n-1}.$$

Since $\langle \gamma_1, \ldots, \gamma_{n-1} \rangle_+ = \Gamma'$ is unmixed and

$$(**) \qquad -c_n\gamma_n \in \Gamma' \text{ with } c_n > 0$$

(see the Proof of 3.1), $b_n < 0$ can occur in $(*)$ for $b_i = 0$ only. Suppose it does, for $i = n-1$, say. We then have

$$-b_n\gamma_n \in -\gamma_{n-1} + \Gamma',$$

and in view of $(**)$ this can occur for finitely many $b_n < 0$ only. Fix one of them.

We then have "relations"

$$\varepsilon_1 = (b_1, \ldots, b_{n-2}, -1, b_n) \in E,$$
$$\varepsilon_2 = (c_1, \ldots, c_{n-2}, c_{n-1}, c_n) \in E,$$

with $b_1, \ldots, b_{n-2} \geq 0$, $b_n < 0$, $c_1, \ldots, c_{n-1} \geq 0$ and $c_n > 0$. Suppose we have $\varepsilon'_1 = (b'_1, \ldots, b'_{n-2}, -1, b_n) \in E$ as well with $b'_1, \ldots, b'_{n-2} \geq 0$ and $\varepsilon'_1 \neq \varepsilon_1$. Then ε'_1 is not in the $\mathbb{Q}$-span of ε_1 and ε_2, and since rank $E = 2$, ε_1 and ε_2 are linearly dependent and hence $b_1 = \cdots = b_{n-2} = 0$, $-\gamma_{n-1} + b_n\gamma_n = 0$ and $b'_1\gamma_1 + \cdots + b'_{n-2}\gamma_{n-2} = 0$ with $b'_i > 0$ for at least one i. We therefore have an invariant monomial $x_1^{b'_1} \cdots x_{n-2}^{b'_{n-2}}$ not divisible by x_n, and x_n is not a nullvariable.

Hence for each $i = 2, \ldots, n-1$ there are only finitely many $\beta = (b_1, \ldots, b_{n-1}) \in (\mathbb{Z}_+)^{n-1}$ such that $(*)$ holds with $b_n < 0$. Moreover, $b_i = 0$ in that case. We can therefore modify h_2 by a G'-homogeneous polynomial in $h_1, h_3 \ldots, h_{n-1}$ to make sure this case does not occur, then do the same successively for $h_3, \ldots, h_{n-1}$. We will then have for $i = 1, \ldots, n-1$

$$(***) \qquad H_i = x_i + \tilde{H}_i \in k[x_1, \ldots, x_{n-1}, x_n]$$

with $\operatorname{ord}_0 \tilde{H}_i > 1$.

Now in $k[G][x_1, \ldots, x_{n-1}]$, the homomorphism $\eta\colon x_i \longmapsto H_i$ is just conjugation by the automorphism $x_i \longmapsto \chi_i x_i$, and it is defined over $k[x_n, x_n^{-1}] \subset k[G]$, with $x_n = \chi_n$. Hence

$$k[x_1, \ldots, x_{n-1}, x_n, x_n^{-1}] = k[H_1, \ldots, H_{n-1}, x_n, x_n^{-1}].$$

Moreover, by $(***)$, $H_1, \ldots, H_{n-1}$ are algebraically independent mod x_n. By 1.10, applied to $B = k[x_n] \subset A$, we have

$$A = k[H_1, \ldots, H_{n-1}, x_n].$$

This finishes the induction.

REFERENCES

[A] S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry*, Tata Institute of Fundamental Research, Lecture Notes on Mathematics and Physics, Bombay (1977).

[AM] S. S. Abhyankar and T.-T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276**(1975), 148–166.

[B-B] A. Bialynicki-Birula, *Remarks on the action of an algebraic torus on k^n, I and II*, Bull. Acad. Pol. Sci. **14**(1966), 177–181 and **15**(1967), 123–125.

[BH] H. Bass and W. Haboush, *Linearizing certain reductive group actions*, Trans. Am. Math. Soc. (2)**292** (1985), 463–481.

[KbR] T. Kambayashi and P. Russell, *On linearizing algebraic torus actions*, J. Pure Appl. Algebra **23**(1982), 243–250.

[KR1] M. Koras and P. Russell, *On linearizing "good"* $\mathbb{C}^*$*-actions on* $\mathbb{C}^3$, Can. Math. Soc. Conf. Proc. **10** (1989), 92–102.

[KR2] ———, *Codimension 2 torus actions on affine n-space*, Canad. Math. Soc. Conf. Proc. **10**(1989), 103–110.

[KR3] ———, *Contractible threefolds and* $\mathbb{C}^*$*-actions on* $\mathbb{C}^3$, manuscript, CICMA preprint series.

[KR4] ———, *On* $\mathbb{C}^3/\mathbb{C}^*$*: the smooth locus is not of hyperbolic type*, in preparation.

[PTD] T. Petrie and T. Tom Dieck, *The Abhyankar-Moh problem in dimension 3*, Springer Lecture Notes in Mathematics **1375**(1989), 48–59.

[RS] P. Russell and A. Sathaye, *On finding and cancelling variables in* $k[X,Y,Z]$, J. Alg. (1)**57**(1979), 151–166.

[S] M. Suzuki, *Propriétés topologiques des polynomes de deux variables complexes et automorphismes algébriques de l'espace* $\mathbb{C}^2$, J. Math. Soc. Japan **26**(1974), 241–257.

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DISCIPLINED SPACES AND CENTRALIZER CLONE SEGMENTS

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ABSTRACT. Our main result implies that for any choice $1 \leq m \leq n \leq p$ of integers there exist finitary algebras A_1 and A_2 that generate the same variety, and such that the initial k-segments of their centralizer clones coincide exactly when $k \leq m$, are isomorphic exactly when $k \leq n$ and are elementarily equivalent exactly when $k \leq p$. The proof uses the existence and properties of disciplined topological spaces which we introduce and investigate here.

1. Introduction.

1.1. In simplest terms, the centralizer clone Clo(A) of a universal algebra A is the category of all homomorphisms between finite powers $A^0 = \{\emptyset\}, A^1 = A, A^2, A^3, \ldots$ of A. For any integer $r \geq 1$, all homomorphisms between the powers $A^0, A^1, \ldots A^r$ form what is called the initial r-segment $\mathrm{Clo}_r(A)$ of Clo(A). It is clear that the equality of centralizer clone segments $\mathrm{Clo}_r(A_1)$ and $\mathrm{Clo}_r(A_2)$ of algebras A_1, A_2 always implies their isomorphism, and that their isomorphism always implies their elementary equivalence. The present paper demonstrates that, subject only to these obvious dependencies, the three upper boundaries of equality, isomorphism and elementary equivalence of centralizer clone segments of universal algebras may be chosen at will.

This result translates an earlier one [7] on continuous maps of metrizable spaces, and exploits the method used by the first author in her preprint [10] to separate equality from elementary equivalence for centralizer clone segments of universal algebras. These two sources, in turn, derive their essence from yet another first author's paper [9] on continuous maps of metrizable topological spaces. The original impetus for [10] came from Ralph McKenzie who suggested that, having completed [9], the first author also investigate centralizer clones of universal algebras.

Results of [7] on continuous maps of metrizable spaces can be 'translated' into our present results on homomorphisms of universal algebras only for metrizable spaces which are disciplined (see 2.2 below). We show that metrizable spaces constructed in [7] are disciplined and have other properties enabling the translation to proceed.

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A somewhat more general notion of a clone is needed for an accurate description of our results.

1.2 . Let $\omega = \{0, 1, 2, \ldots\}$ denote the set consisting of all finite ordinals $n = \{0, \ldots, n-1\}$.

A subcategory T of the category Set of all sets and maps is called a *clone on a non-empty set* X if the set obj T of all objects of T consists of all finite Cartesian powers $X^0 = \{\emptyset\}, X^1 = X, X^2 = X \times X, \ldots$ of X, the product projections $p_i^{(m)}: X^m \longrightarrow X$ with $i \in m$ are morphisms in T for any $m \in \omega$, and every $X^m \in$ obj T accompanied by all Cartesian product projections $p_i^{(m)}: X^m \longrightarrow X$ with $i \in m$ is a categorical product, in T, of m copies of X. This means that for any $m, n \in \omega$ and any m-tuple $f_i: X^n \longrightarrow X$ of maps in T, the map $f: X^n \longrightarrow X^m$ defined by

$$f(y) = \big(f_0(y), \ldots, f_{m-1}(y)\big) \text{ for all } y \in X^n$$

also belongs to T. In particular, for each $\psi: m \longrightarrow n$ with $m, n \in \omega$, the category T contains the map $p^{[\psi]}: X^n \longrightarrow X^m$ given for every $y = (y_0, \ldots, y_{n-1}) \in X^n$ by

$$p^{[\psi]}(y_0, \ldots, y_{n-1}) = (y_{\psi(0)}, \ldots, y_{\psi(m-1)}).$$

Set $\Gamma_T(m) = X^m$ for any $m \in \omega$, and $\Gamma_T(\psi) = p^{[\psi]}$ for any mapping $\psi: m \longrightarrow n$ with $m, n \in \omega$. Then $\Gamma_T: \mathrm{Ord}_\omega \longrightarrow T$ is a well-defined contravariant functor from the category Ord_ω of all finite ordinals and all mappings between them, and $\Gamma_T(\psi_j^{(n)}) = p_j^{(n)}$ for every map $\psi_j^{(n)}: 1 \longrightarrow n$ with $\psi_j^{(n)}(0) = j \in n$.

Let k be an abstract category and let H be an isomorphism of k onto a clone T on a non-empty set X. Then the composite $\alpha = H^{-1} \circ \Gamma_T : \mathrm{Ord}_\omega \longrightarrow k$ is a contravariant functor which endows k with the following additional structure:

(g) k has a uniquely determined object a such that obj $k = \{a^n \mid n \in \omega\}$, where a^n is an n-th power of a in the category k, and

(e) k has a specific enumeration of product projections $\pi_j^{(n)}: a^n \longrightarrow a$ by members of n.

Indeed, it suffices to set $a = \alpha(1)$, $a^n = \alpha(n)$ and $\pi_j^{(n)} = \alpha(\psi_j^{(n)})$ for $j \in n \in \omega$. Conversely, the information provided in (g) and (e) gives rise to such a contravariant functor $\alpha: \mathrm{Ord}_\omega \longrightarrow k$.

A category k for which (g) and (e) hold (or, equivalently, a category accompanied by such a functor $\alpha: \mathrm{Ord}_\omega \longrightarrow k$) will be called an *abstract clone* and $a \in$ obj k its *generating object*.

Therefore, up to trivial cases when α is not faithful, any abstract clone is the categorical dual of an algebraic theory as defined by Lawvere.

We say that abstract clones k and k' are *isomorphic*, and we write

$$k \simeq k',$$

if there exists an isofunctor Φ of k onto k' such that $\Phi \circ \alpha = \alpha'$ for the respective contravariant functors α and α' of Ord_ω into k and k'. In particular, any isomorphism preserves generating objects and projection enumeration.

1.3 . A pair (k, F) is called a *clone* whenever k is an abstract clone and $F\colon k \longrightarrow \mathrm{Set}$ is a faithful functor which preserves finite products.

We say that clones (k, F) and (k', F') are *isomorphic*, and we write

$$(k, F) \simeq (k', F'),$$

whenever the abstract clones k and k' are isomorphic. We say that the two clones are *equal*, and we write

$$(k, F) = (k', F'),$$

if there exists an isomorphism Φ of the abstract clone k onto the abstract clone k' (that is, an isofunctor Φ such that $\Phi \circ \alpha = \alpha'$) for which $F' \circ \Phi = F$.

1.4 . If $\mathcal{K}$ is a category with finite products, then every object a of $\mathcal{K}$ determines an abstract clone k, uniquely up to an isomorphism, as follows: for every $n \in \omega$ we select an n-th power a^n of a and an enumeration of its product projections, and then let k be the full subcategory of $\mathcal{K}$ generated by these powers.

If $(\mathcal{K}, \mathcal{U})$ is a concrete category such that $\mathcal{K}$ has finite products and the forgetful functor $U\colon \mathcal{K} \longrightarrow \mathrm{Set}$ preserves them (that is, if $(\mathcal{K}, U)$ has finite concrete products), then every object a of $\mathcal{K}$ with $U(a) \neq \emptyset$ determines, up to an equality of clones induced on the non-empty set $U(a)$, a clone (k, F), namely that consisting of an abstract clone k determined by a in $\mathcal{K}$ and the restriction F of U to k. Any such clone (k, F) will be denoted as $\mathrm{Clo}(a)$, or $\mathrm{Clo}(a, \mathcal{K})$ when $\mathcal{K}$ needs to be explicitly mentioned.

1.5 . Any nonvoid universal algebra A of a finitary type Σ determines two distinct clones on its underlying set X. As suggested by [6], for instance, these are:

(1) the clone $\mathrm{Clo}(A)$ formed by all homomorphisms between finite powers of A, called the *centralizer clone* of A, and

(2) the *operation clone* of A, which is the smallest clone on X containing all operations of A.

While the operation clone of A determines the centralizer clone $\mathrm{Clo}(A)$ uniquely, its abstract clone does not do this at all, as shown in Theorem 1.9 below.

1.6 . Let $r \geq 0$ be an integer. If k is an abstract clone generated by the object a, then its full subcategory generated by $\{a^j \mid j = 0, 1, \ldots, r\} \subset \mathrm{obj}\, k$ augmented by the enumeration of the product projections $\pi_i^{(j)}\colon a^j \longrightarrow a$ inherited from k is called the *r-segment* of k. A pair (k_r, F_r) is the r-segment of a clone (k, F) if k_r is the r-segment of k and F_r is the restriction of F to the full subcategory k_r of k. For any concrete category $(\mathcal{K}, U)$ with finite concrete products and any $a \in \mathrm{obj}\, \mathcal{K}$ with $U(a) \neq \emptyset$, the r-segment of a clone $\mathrm{Clo}(a)$ will be denoted by $\mathrm{Clo}_r(a)$.

We say that two r-segments k_r, k'_r of abstract clones k, k' are isomorphic and write

$$k_r \simeq k'_r$$

if there exists an isofunctor Φ_r of k_r onto k'_r such that $\Phi_r \circ \alpha_r = \alpha'_r$ where α_r, α'_r are restrictions of the respective contravariant functors α and α'.

If (k_r, F_r) and (k'_r, F'_r) are r-segments of clones, we say that they are *isomorphic* and write

$$(k_r, F_r) \simeq (k'_r, F'_r)$$

whenever $k_r \simeq k'_r$. Finally, the two clone r-segments (k_r, F_r) and (k'_r, F'_r) are *equal*, and we write

$$(k_r, F_r) = (k'_r, F'_r)$$

if there exists an isofunctor Φ_r of k_r onto k'_r such that $\Phi_r \circ \alpha_r = \alpha'_r$ and $F'_r \circ \Phi_r = F_r$.

1.7. In addition to their equality and isomorphism, we shall also investigate elementary equivalence of clones and clone segments. We say that clones or r-segments of clones (k, F) and (k', F') are *elementarily equivalent*, and write

$$(k, F) \approx (k', F'),$$

if the abstract clones or r-segments of abstract clones k and k' are elementarily equivalent (in symbols $k \approx k'$), that is, whenever k and k' satisfy the same formulas of the first order language of clone theory or of its fragment appropriate to r-segments. Paragraph 1.8 below introduces this language, as used by Taylor [8], and then in [9] and [7].

1.8. Any abstract clone k with a generating object a can be viewed as an ω-sorted universal algebra whose carrier X_n of the n-th sort is the set $k(a^n, a)$ of all k-morphisms $a^n \longrightarrow a$, and which has

(c) n distinct nullary operations $c_0^{(n)}, \ldots, c_{n-1}^{(n)}$ of each sort $n \in \omega$, and

(s) for any $m, n \in \omega$, an operation S_m^n of heterogeneous arity $(n, \overbrace{m, \ldots, m}^{n \text{ times}})$ whose values are of the sort m, that is,

$$S_m^n : X_n \times \overbrace{X_m \times \cdots \times X_m}^{n \text{ times}} \longrightarrow X_m.$$

Indeed, we need only interpret $c_0^{(n)}, \ldots, c_{n-1}^{(n)}$ as the product projections $\pi_0^{(n)}, \ldots, \pi_{n-1}^{(n)}$ of k and write

$$S_m^n(h, f_0, \ldots, f_{n-1}) = h \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1})$$

in terms of the composition in k—where $f_0 \dot{\times} \cdots \dot{\times} f_{n-1}$ is the unique k-morphism f in $k(a^m, a^n)$ for which $\pi_i^{(n)} \circ f = f_i$ for all $i \in n$. These operations also satisfy all equations below, namely,

(RU) $S_n^n(h, c_0^{(n)}, \ldots, c_{n-1}^{(n)}) = h$ for every $n \in \omega$,

(LU) $S_m^n(c_i^{(n)}, f_0, \ldots, f_{n-1}) = f_i$ for all $m, n \in \omega$ and $i \in n$,

(AC) $S_m^p\big(h, S_m^n(g_0, f_0, \ldots, f_{n-1}), \ldots, S_m^n(g_{p-1}, f_0, \ldots, f_{n-1})\big) = S_m^n\big(S_n^p(h, g_0, \ldots, g_{p-1}), f_0, \ldots, f_{n-1}\big)$ for all $m, n, p \in \omega$.

Hence every abstract clone k determines a unique ω-sorted algebra whose operations are described in (c) and (s), and which satisfies equations (RU), (LU) and (AC). Conversely, any such ω-sorted algebra determines, up to an isomorphism, an abstract clone k.

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To see this, we choose an abstract generating object a for k and then formally require that $k(a^m, a^n)$ consist of all n-tuples of members of the carrier X_m of the sort m [with the unique 'empty' 0-tuple in case of $n = 0$]. If a 'composite' $g \circ f$ of $f = (f_0, \dots, f_{n-1}) \in k(a^m, a^n)$ and $g = (g_0, \dots, g_{p-1}) \in k(a^n, a^p)$ is defined by

$$g \circ f = \big(S^n_m(g_0, f_0, \dots, f_{n-1}), \dots, S^n_m(g_{p-1}, f_0, \dots, f_{n-1})\big),$$

then the associativity of $\circ$ follows from (AC). The n-tuple $(c^{(n)}_0, \dots, c^{(n)}_{n-1})$ is the unit in $k(a^n, a^n)$, and a^n is the n-th power of a with the projections $c^{(n)}_0, \dots, c^{(n)}_{n-1} \in X_n = k(a^n, a)$ because of (RU) and (LU).

The first order language of clone theory is nothing else but the first order language of the ω-sorted algebras described above. Let $r \geq 1$ be an integer. Having restricted the sorts to $\{0, 1, \dots, r\}$ and the operations to those $c^{(n)}_i$ and S^n_m for which $i \in n$ and $m, n \leq r$, we obtain a fragment of the first order language of clones which is just the first order language of clone r-segments.

Given an abstract clone or an r-segment of an abstract clone k, we shall call the above ω-sorted or $(r+1)$-sorted algebra the *clone algebra* of k. The carrier of its n-th sort, that is, the set $k(a^n, a)$ of k-morphisms, will be denoted by $(k)_n$.

It is clear that two abstract clones or clone segments are isomorphic exactly when their corresponding clone algebras are isomorphic and, more generally, that product preserving functors that also preserve product projection enumeration correspond exactly to homomorphisms of the corresponding clone algebras.

1.9. MAIN THEOREM. *If $m, n, p \in \{1, 2, \dots, \infty\}$ are such that $m \leq n \leq p$, then there exist finitary algebras A_1 and A_2 whose centralizer clones* $\mathrm{Clo}(A_1)$ *and* $\mathrm{Clo}(A_2)$ *satisfy*

$$\begin{aligned} m &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(A_1) = \mathrm{Clo}_i(A_2)\}, \\ n &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(A_1) \simeq \mathrm{Clo}_i(A_2)\}, \\ p &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(A_1) \approx \mathrm{Clo}_i(A_2)\}. \end{aligned}$$

The algebras A_1, A_2 have isomorphic operation clones.

Next we outline the contents of the paper. In the second section, we assign a finitary algebra $\mathbb{A}(X)$, called a *trace* of X, to any metrizable topological space X. We also introduce the notion of a *disciplined* (metrizable) space X, for which we demonstrate a close connection between continuous maps $X^n \longrightarrow X$ and homomorphisms $\big(\mathbb{A}(X)\big)^n \longrightarrow \mathbb{A}(X)$. Then (see 2.9) we deduce that, for any two infinite disciplined spaces X_1, X_2,

$$\mathrm{Clo}_k(X_1) = \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) = \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

In the third section (see 3.5), we prove that

$$\mathrm{Clo}_k(X_1) \simeq \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \simeq \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$$

for any infinite disciplined spaces X_1, X_2 with no *absolute constants*—a clone notion introduced in 3.2. In the fourth section on elementary equivalence (see 4.1), we show that such spaces also satisfy

$$\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

In [7], we presented a general construction method that produces metric spaces whose clones of continuous maps have certain useful properties, and then applied it to obtain a result similar to the present Theorem 1.9:

> [7] for any choice of $m, n, p \in \{1, 2, \ldots, \infty\}$ with $m \leq n \leq p$, there exist metric spaces X_1 and X_2 such that
>
> $$\begin{aligned} m &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) = \mathrm{Clo}_i(X_2)\}, \\ n &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \simeq \mathrm{Clo}_i(X_2)\}, \\ p &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \approx \mathrm{Clo}_i(X_2)\}. \end{aligned}$$

In the fifth section, we briefly describe this method, and then supplement [7] by showing that all spaces it produces are disciplined, and almost none of them have absolute constants. Together with the results of Sections 2–4 mentioned earlier, this already shows that the algebras $A_1 = \mathbb{A}(X_1)$ and $A_2 = \mathbb{A}(X_2)$ satisfy the first statement of Theorem 1.9. Its remainder—the claim that the operation clones of A_1, A_2 are isomorphic—is proved in the sixth section, where we show that traces of any two infinite metrizable spaces satisfy precisely the same set of identities. This implies that operation clones of algebras $A_1 = A_1(m,n,p)$ and $A_2 = A_2(m,n,p)$ are isomorphic for all choices $m \leq n \leq p$ in $\{1, 2, \ldots, \infty\}$, so that they determine the same abstract clone. Moreover, all these algebras will be constructed over the same underlying set. This means that an abstract (operation) clone can have representations on *the same set* which vary to the extent that their centralizer clones realize *every* triple $m \leq n \leq p$ in $\{1, 2, \ldots, \infty\}$, in the sense described by Theorem 1.9.

Using [7] as a source of disciplined spaces without absolute constants, in the seventh section we apply the trace to 'translate' other results of [7] into results on finitary algebras. This section also describes alternate traces (unary and infinitary), and restates Theorem 1.9 for a finitary algebra and its reduct.

2. **Disciplined spaces and their algebraic traces.**

2.1. To any metrizable topological space $X = (P, t)$ we now assign its *algebraic trace* $\mathbb{A}(X)$. This is a universal algebra $\mathbb{A}(X) = (Q, \{\cdot, \gamma\} \cup \{\pi_k \mid k \in \omega\})$ with a single binary operation $\cdot$ and countably many unary operations $\pi_0, \pi_1, \ldots$ and γ. The underlying set Q of $\mathbb{A}(X)$ is the disjoint union

$$Q = P \cup S \cup \{\lambda\},$$

where the set S consists of all one-to-one sequences $\{p_k \mid k \in \omega\}$ of members of P. The operations of $\mathbb{A}(X)$ are defined as follows:

$$q \cdot q' = \begin{cases} q & \text{if } q = q' \in P \\ \lambda & \text{otherwise;} \end{cases}$$

for every $k \in \omega$, we set

$$\pi_k(q) = \begin{cases} p_k & \text{if } q = \{p_k \mid k \in \omega\} \in S \\ q & \text{for every } q \in P \cup \{\lambda\}; \end{cases}$$

and finally, the operation γ which traces the topology t of X is given by

$$\gamma(q) = \begin{cases} p & \text{if } q \in S \text{ and } q \text{ converges to } p \text{ in } X \\ \lambda & \text{if } q \in S \text{ does not converge in } X \\ q & \text{for every } q \in P \cup \{\lambda\}. \end{cases}$$

It is clear that $\mathbb{A}(X)$ is isomorphic to $\mathbb{A}(X')$ whenever X is homeomorphic to X'. Conversely, if the traces $\mathbb{A}(X)$, $\mathbb{A}(X')$ are isomorphic then X is homeomorphic to X'. Indeed, if $\mathbb{A}(X)$ is a singleton then $X = X' = \emptyset$. Else $P \subset Q$ consists of all $q \in Q$ with $q \cdot q = q$ satisfying $q \cdot q' \neq q$ for every $q' \neq q$, and hence P is bijective to the underlying set P' of X', and X is homeomorphic to X' because of the remaining unary operations.

2.2. DEFINITION. Let $n \in \omega$. We say that a topological space $X = (P, t)$ is *n-disciplined* if every continuous map $f\colon X^n \longrightarrow X$ decomposes as a projection followed by a homeomorphism onto a closed subset of X or, more precisely, if for some set $M \subseteq n = \{0, \ldots, n-1\}$ and some homeomorhism $g\colon X^M \longrightarrow X$ of X^M onto a closed subset of X,

$$f = g \circ \pi^{(M)},$$

where $\pi^{(M)}\colon X^n \longrightarrow X^M$ is the projection of X^n onto X^M associated with the inclusion map $M \subseteq n$.

A space X is *disciplined* if it is n-disciplined for every $n \in \omega$.

REMARKS. (a) It is clear that every T_1-space is 0-disciplined.

(b) A space X is n-disciplined if and only if for every continuous map $f\colon X^n \longrightarrow X$ there exist an integer $k \geq 0$, a one-to-one mapping $\psi\colon k \longrightarrow n$, and a homeomorphism $g\colon X^k \longrightarrow X$ onto a closed subset of X such that $f = g \circ \pi^{[\psi]}$, where $\pi^{[\psi]}\colon X^n \longrightarrow X^k$ is the (surjective) projection given by $\pi^{[\psi]}(\varphi) = \varphi \circ \psi$ for every $\varphi \in X^n$. Any such decomposition $f = g \circ \pi^{[\psi]}$ will be called *standard*.

For any integer $n \geq 0$, let $\lambda^{(n)}\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ be the constant map whose value is λ. Here is the central result of this section.

2.3. PROPOSITION. *Let $\mathbb{A}(X)$ be the algebraic trace of a disciplined metrizable space $X = (P, t)$, and let $n \geq 0$. Then*

(1) every continuous map $f\colon X^n \longrightarrow X$ has a unique extension to a homomorphism $E(f)\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$, and $E(f)$ is constant if and only if f is constant,

(2) if $h\colon \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ *is a homomorphism, then either* $h = \lambda^{(n)}$, *or else* $h(P^n) \subseteq P$ *and the restriction of* h *to* P^n *maps* X^n *continuously into* X.

We recall that the algebraic trace $\mathbb{A}(X) = (Q, \{\cdot, \gamma\} \cup \{\pi_j \mid j \in \omega\})$ of a space $X = (P, \iota)$ is defined on the disjoint union $Q = P \cup S \cup \{\lambda\}$. For $k \geq 1$ and an arbitrary one-to-one map $g\colon P^k \longrightarrow P$ we define a map

$$h_g\colon Q^k \longrightarrow Q$$

as follows. For $q = (q_0, \ldots, q_{k-1}) \in Q^k$, we set

$$\text{(h1)} \qquad h_g(q_0, \ldots, q_{k-1}) = \begin{cases} \lambda & \text{if at least one } q_i = \lambda \\ g(q) & \text{if } q \in P^k. \end{cases}$$

The case still left undefined is that of a k-tuple $q \in (P \cup S)^k$ with $q_l \in S$ for at least one $l \in k$. For any such k-tuple $(q_0, \ldots, q_{k-1})$ and every $i \in k$ and $j \in \omega$ we first write

$$\tilde{q}_{j,i} = \begin{cases} q_i & \text{when } q_i \in P \\ \pi_j(q_i) & \text{when } q_i \in S, \end{cases}$$

and then assemble a sequence $\{\tilde{q}_j \mid j \in \omega\}$ with the members $\tilde{q}_j = (\tilde{q}_{j,0}, \ldots, \tilde{q}_{j,k-1}) \in P^k \subset Q^k$. This sequence is one-to-one because $q_l \in S$ for some $l \in k$. Since the mapping g is one-to-one,

$$\text{(h2)} \qquad h_g(q_0, \ldots, q_{k-1}) = \{g(\tilde{q}_j) \mid j \in \omega\}$$

is a one-to-one sequence of elements of P. Whence $h_g(q_0, \ldots, q_{k-1}) \in S$, and (h2) and (h1) define a mapping $h_g\colon Q^k \longrightarrow Q$ correctly and completely for any $k \geq 1$.

When $k = 0$, we have $P^0 = Q^0 = \{\emptyset\}$ and, for any $g\colon P^0 \longrightarrow P$, we define $h_g\colon Q^0 \longrightarrow Q$ to be the map with $h_g(\emptyset) = g(\emptyset)$.

Proposition 2.3 is proved in 2.4–2.9 below.

2.4. LEMMA. *Let* $k \geq 0$ *be an integer and let* $g\colon P^k \longrightarrow P$ *be a homeomorphism of* X^k *onto a closed subset of* X. *Then* $E(g) = h_g\colon \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ *is a homomorphism.*

PROOF. Write $h_g = h$. Since $\{p\}$ is a subalgebra of $\mathbb{A}(X)$ for every $p \in P$, we need consider only the case of $k \geq 1$. Let $q = (q_0, \ldots, q_{k-1})$ and $q' = (q'_0, \ldots, q'_{k-1})$ belong to Q^k.

a) First we show that the binary operation $\cdot$ is preserved by h.

a1) If $q \neq q'$, then $q_l \cdot q'_l = \lambda$ for some $i \in k$, so that $h(q \cdot q') = \lambda$. We show that $h(q) \cdot h(q') = \lambda$ as well. If $q_l \in S \cup \{\lambda\}$ for some $l \in k$, then $h(q) \in S \cup \{\lambda\}$ and hence $h(q) \cdot z = \lambda$ for all $z \in Q$. Since the operation $\cdot$ is commutative, in the only remaining case we have $q, q' \in P^k$. But h coincides with g on P^k and g is one-to-one, so that $h(q) \neq h(q')$ and $h(g) \cdot h(g') = \lambda$ follows.

a2) If $q = q'$, then either $q_l \in S \cup \{\lambda\}$ for some $l \in k$ and $h(q \cdot q') = \lambda = h(q) \cdot h(q')$ as in the previous case, or $q \in P^k$, and hence $q \cdot q = q$. But then $h(q \cdot q) = g(q)$ and $h(q) \cdot h(q) = g(q) \cdot g(q) = g(q)$.

b) Next we prove that h preserves π_j for each $j \in k$.

b1) If $q \in (P \cup \{\lambda\})^k$, then $\pi_j(q_i) = q_i$ for all $i \in k$, and hence $\pi_j(q) = q$. Also, $h(q) \in P \cup \{\lambda\}$ by (h1), and $\pi_j\big(h(q)\big) = h(q) = h\big(\pi_j(q)\big)$ follows.

b2) If $q_l = \lambda$ for some $l \in k$, then $\pi_j(q_l) = \lambda$, so that $h\big(\pi_j(q)\big) = \lambda = h(q) = \pi_j\big(h(q)\big)$, by (h1) again.

b3) In the remaining case of a $q \in (P \cup S)^k$ with $q_l \in S$ for some $l \in k$, the k-tuple $r = \big(\pi_j(q_0), \dots, \pi_j(q_{k-1})\big)$ is such that $\pi_j(q_l) = q_l$ when $q_l \in P$, and $\pi_j(q_l) \in P$ is the j-th member of the sequence q_l when $q_l \in S$, so that $r \in P^k$ and $h(r) = g(r) = \pi_j\big(h(q)\big)$ by (h2).

c) Finally, we show that $h\big(\gamma(q_0), \dots \gamma(q_{k-1})\big) = \gamma\big(h(q)\big)$ for all $q = (q_0, \dots, q_{k-1}) \in Q^k$. This is clearly true for $q \in (P \cup \{\lambda\})^k$ because γ is the identity on $P \cup \{\lambda\}$ and $h(q) \in P \cup \{\lambda\}$ for any such q, and also for any k-tuple q with some $q_l = \lambda$, for then $h\big(\gamma(q_0), \dots, \gamma(q_{k-1})\big) = \lambda = \gamma(\lambda) = \gamma\big(h(q)\big)$ by (h1).

In the remaining case, we have a $q \in (P \cup S)^k$ with $q_l \in S$ for some $l \in k$. As in the clause (h2) of the definition of h, let $\tilde{q} = \{\tilde{q}_j \mid j \in \omega\}$ be the one-to-one sequence of elements $\tilde{q}_j = (\tilde{q}_{j,0}, \dots, \tilde{q}_{j,k-1})$ of P^k, such that $\tilde{q}_{j,i} = q_i$ when $q_i \in P$, and $\tilde{q}_{j,i}$ is the j-th member of q_i when $q_i \in S$. First, let us suppose that $\tilde{q}$ converges to a point $r = (r_0, \dots, r_{k-1})$ in the space $X^k = (P, t)^k$. Then, for every $i \in k$, the sequence $\{\tilde{q}_{j,i} \mid j \in \omega\}$ converges to r_i in X, so that $\big(\gamma(q_0), \dots, \gamma(q_{k-1})\big) = r$, and hence $h\big(\gamma(q_0), \dots, \gamma(q_{k-1})\big) = h(r)$. On the other hand, $h(q) = \{g(\tilde{q}_j) \mid j \in \omega\} \in S$ by (h2), and the sequence $\{g(\tilde{q}_j) \mid j \in \omega\}$ converges to $g(r)$ because g is continuous. But $h(r) = g(r)$ since $r \in P^k$, and therefore $\gamma\big(h(q)\big) = h(r) = h\big(\gamma(q_0), \dots, \gamma(q_{k-1})\big)$. Secondly, let us suppose that $\tilde{q}$ does not converge in X^k. Then there must be a sequence $\{\tilde{q}_{j,i} \mid j \in \omega\}$ which does not converge in X. Whence $\gamma\{\tilde{q}_{j,i} \mid j \in \omega\} = \lambda$. Simultaneously, since g is a homeomorphism of P^k onto a closed subset of (P, t), the sequence $\{g(\tilde{q}_j) \mid j \in \omega\} = h(q)$ cannot converge in (P, t) either. Therefore $\gamma\big(h(q)\big) = \lambda = h\big(\gamma(q_0), \dots, \gamma(q_{k-1})\big)$. ■

2.5. LEMMA. *Let $k \geq 0$ be an integer and let $h \colon \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ be a homomorphism. Then either h is constant with the value in $P \cup \{\lambda\}$, or else $h(\lambda, \dots, \lambda) = \lambda$, $h(P^k) \subseteq P$, and the restriction of h to P^k is not constant.*

PROOF. (a) Since $z \cdot \lambda = \lambda$ for all $z \in Q$, for every $q \in Q^k$ we have $q \cdot (\lambda, \dots, \lambda) = (\lambda, \dots, \lambda)$, and hence also $h(q) \cdot h(\lambda, \dots, \lambda) = h(\lambda, \dots, \lambda)$. If $h(\lambda, \dots, \lambda) \neq \lambda$, then $h(\lambda, \dots, \lambda) \in P$ and hence $h(q) = h(\lambda, \dots, \lambda)$ for all $q \in Q^k$. Therefore $h(\lambda, \dots, \lambda) = \lambda$ for any h which is not a constant whose value belongs to P.

(b) Suppose that $p = (p_0, \dots, p_{k-1}) \in P^k$ and $h(p) = \lambda$. We aim to show that h must be the constant with the value λ. Since $\gamma(z) = z$ exactly when $z \in P \cup \{\lambda\}$, the homomorphism h must map $(P \cup \{\lambda\})^k$ into $P \cup \{\lambda\}$. First we show that $h(p') = \lambda$ for any other $p' = (p'_0, \dots, p'_{k-1}) \in P^k$. For $i \in k$, we set $q_i = p_i$ when $p'_i = p_i$ and, when $p'_i \neq p_i$, choose $q_i \in S$ with $\pi_0(q_i) = p_i$ and $\pi_1(q_i) = p'_i$. Then $\big(\pi_0(q_0), \dots, \pi_0(q_{k-1})\big) = p$ and $\big(\pi_1(q_0), \dots, \pi_1(q_{k-1})\big) = p'$. Whence $\pi_0\big(h(q_0, \dots, q_{k-1})\big) = h\big(\pi_0(q_0), \dots, \pi_0(q_{k-1})\big) = h(p) = \lambda$. But then $h(q_0, \dots, q_{k-1}) = \lambda$ because $\pi_0^{-1}\{\lambda\} = \{\lambda\}$, and $h(p') = \pi_1\big(h(q_0, \dots, q_{k-1})\big) = \pi_1(\lambda) = \lambda$ follows. Therefore $h(P^k) = \{\lambda\}$. Next we show that $h(q) = \lambda$ for every $q = (q_0, \dots, q_{k-1}) \in (P \cup \{\lambda\})^k$. For any such q, there exist

$p = (p_0, \dots, p_{k-1})$ and $p' = (p'_0, \dots, p'_{k-1})$ in P^k with $q_i = p_i \cdot p'_i$ for every $i \in k$, and hence $h(q) = h(p) \cdot h(p') = \lambda \cdot \lambda = \lambda$. Finally, for any $q = (q_0, \dots, q_{k-1}) \in Q^k$ we have $\big(\pi_0(q_0), \dots, \pi_0(q_{k-1})\big) \in (P \cup \{\lambda\})^k$, so that $\pi_0\big(h(q)\big) = h\big(\pi_0(q_0), \dots, \pi_0(q_{k-1})\big) = \lambda$. Since $\pi_0(z) = \lambda$ in $\mathbb{A}(X)$ only when $z = \lambda$, this proves that $h(q) = \lambda$. Whence h is the constant with the value λ.

If h is not constant, then (a) and (b) show that $h(\lambda, \dots, \lambda) = \lambda$, $h(P^k) \subseteq P$, and that the restriction of h to P^k is not constant. ∎

2.6. LEMMA. *Let $k \geq 0$ be an integer and let $h_1, h_2: \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ be homomorphisms such that $h_1(p) = h_2(p)$ for every $p \in P^k$. Then $h_1 = h_2$.*

PROOF. Since the claim is trivial when $k = 0$, let us assume that $k \geq 1$. By Lemma 2.5, the homomorphisms h_1, h_2 agree on $P^k \cup \{\lambda\}^k$.

(a) To show that $h_1(q) = h_2(q)$ for each $q = (q_0, \dots, q_{k-1}) \in (P \cup \{\lambda\})^k$, we select $p = (p_0, \dots, p_{k-1})$ and $p' = (p'_0, \dots, p'_{k-1})$ in P^k so that $p_i \cdot p'_i = q_i$ for every $i \in k$, and conclude that $h_1(q) = h_1(p) \cdot h_1(p') = h_2(p) \cdot h_2(p') = h_2(q)$.

(b) Let $q = (q_0, \dots, q_{k-1}) \in Q^k$. Then $\big(\pi_j(q_0), \dots, \pi_j(q_{k-1})\big) \in (P \cup \{\lambda\})^k$ for every $j \in \omega$. The maps h_1, h_2 satisfy $\pi_j\big(h_1(q)\big) = h_1\big(\pi_j(q_0), \dots, \pi_j(q_{k-1})\big)$ and $\pi_j\big(h_2(q)\big) = h_2\big(\pi_j(q_0), \dots, \pi_j(q_{k-1})\big)$. But then $\pi_j\big(h_1(q)\big) = \pi_j\big(h_2(q)\big)$ for every $j \in \omega$ because h_1, h_2 agree on $(P \cup \{\lambda\})^k$, and $h_1(q) = h_2(q)$ follows. ∎

2.7. Now we are in a position to prove Proposition 2.3(1). Let $X = (P, t)$ be a disciplined metrizable space, and let $f: X^n \longrightarrow X$ be continuous with $n \geq 0$. Since X is disciplined, we have a standard decomposition $f = g \circ \pi^{[\psi]}$ of f, where $\pi^{[\psi]}: X^n \longrightarrow X^k$ for some $k \geq 0$ and an injective $\psi: k \longrightarrow n$, and $g: X^k \longrightarrow X$ is a homeomorphism of X^k onto a closed subset of X. Then the homomorphism $h_g: \mathbb{A}(X)^k \longrightarrow \mathbb{A}(X)$ from Lemma 2.4 extends g. Whence $E(f) = h_g \circ \pi^{[\psi]}$, where $\pi^{[\psi]}$ denotes the projection $\mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)^k$ associated with ψ, and $E(f)$ is a homomorphism extending f. Its uniqueness follows by Lemma 2.6. The homomorphism $E(f)$ is constant exactly when f is constant, by Lemma 2.5. This completes the proof of Proposition 2.3(1).

Proposition 2.3(2) will follow directly from Lemma 2.6 and the Lemma 2.8 below.

2.8. LEMMA. *Let $n \geq 0$ be an integer, and let $X = (P, t)$ be a metrizable space. Then any homomorphism $h: \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ other than $\lambda^{(n)}$ is an extension of a continuous map $f: X^n \longrightarrow X$.*

PROOF. If $h \neq \lambda^{(n)}$ then, by Lemma 2.5, either h is a constant whose value belongs to P, or else $h(\lambda, \dots, \lambda) = \lambda$ and $h(P^n) \subseteq P$. Thus we need consider only the latter case for $n \geq 1$.

Suppose that the restriction $f: X^n \longrightarrow X$ of h to P^n is not continuous at some $r = (r_0, \dots, r_{n-1}) \in P^n$. Then there exists a sequence $\{v_j \mid j \in \omega\}$ which converges to r in X^n and such that, for some $\epsilon > 0$, the distance from $f(v_j)$ to $f(r)$ is never less than ϵ. If we write $v_j = (v_{j,0}, \dots v_{j,n-1})$ for each $j \in \omega$, then the sequence $v^i = \{v_{j,i} \mid j \in \omega\}$ converges to r_i for every $i \in n$. We need to produce a sequence $\{w_j \mid j \in \omega\}$ in P^n amenable to algebraic arguments, that is, a sequence whose coordinate sequences $\{w_{j,i} \mid j \in \omega\}$ are either one-to-one or constant for $i \in n$. We proceed as follows. Any sequence contains a subsequence that is either constant or one-to-one. There exists a subsequence $\{v_j^0 \mid j \in \omega\}$ of $\{v_j \mid j \in \omega\}$ whose first-coordinate sequence $\{v_{j,0}^0 \mid j \in \omega\}$ is one-to-one or constant. Then we find a subsequence $\{v_j^1 \mid j \in \omega\}$ of $\{v_j^0 \mid j \in \omega\}$ such that the sequence $\{v_{j,1}^1 \mid j \in \omega\}$ of its second coordinates is one-to-one or constant, and repeat this process until, having exhausted all n coordinates, we arrive at a subsequence $\{w_j \mid j \in \omega\}$ of $\{v_j \mid j \in \omega\}$. If $w_j = (w_{j,0}, \ldots, w_{j,n-1})$ and $i \in n$, then the i-th coordinate sequence $w^i = \{w_{j,i} \mid j \in \omega\}$ is either constant with $w_{j,i} = r_i$ for all $j \in \omega$, or else it is one-to-one and converges to r_i.

For each $i \in n$, we choose $q_i \in Q$ as follows:

$$q_i = \begin{cases} w^i & \text{if } w^i \text{ is one-to-one} \\ r_i & \text{if } w^i \text{ is constant.} \end{cases}$$

Then $q_i \in S$ in the first instance, $q_i \in P$ in the second and, moreover, $\gamma(q_i) = r_i$ and $\pi_j(q_i) = w_{j,i}$ for all $i \in n$ and $j \in \omega$. Thus, in particular, $q = (q_0, \ldots, q_{n-1}) \in (P \cup S)^n$ and $\big(\gamma(q_0), \ldots, \gamma(q_{n-1})\big) = r$ in $\mathbb{A}(X)^n$. Also, for every $j \in \omega$,

$$\pi_j\big(h(q)\big) = h\big(\pi_j(q_0), \ldots, \pi_j(q_{n-1})\big) = h(w_{j,0}, \ldots, w_{j,n-1}) = f(w_j) \in P$$

because $h(P^n) \subseteq P$. From $\pi_j^{-1}(P) = P \cup S$ it follows that either $h(q) \in P$ or $h(q) \in S$. In the first case $h(q) = f(w_j)$ for all $j \in \omega$, while $h(q) = \{f(w_j) \mid j \in \omega\} \in S$ in the second. But then

$$\gamma\big(h(q)\big) = h\big(\gamma(q_0), \ldots, \gamma(q_{n-1})\big) = h(r) = f(r),$$

and this is possible only when either $f(w_j) = f(r)$ for all $j \in \omega$, or $\{f(w_j) \mid j \in \omega\}$ is one-to-one and converges to $f(r)$. But this is impossible since $\{f(w_j) \mid j \in \omega\}$ is a subsequence of the sequence $\{f(v_j) \mid j \in \omega\}$ which does not converge to r. ∎

The proof of Proposition 2.3 is now complete.

2.9. PROPOSITION. *Let $k \geq 0$ be an integer and let X_1 and X_2 be disciplined metrizable spaces. Then*

$$\mathrm{Clo}_k(X_1) = \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) = \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

PROOF. This is an immediate consequence of Proposition 2.3. ∎

3. Absolute constants in disciplined spaces.

3.1 . Our Theorem 1.9 speaks about the equality $=$, isomorphism $\simeq$ and elementary equivalence $\approx$ of centralizer clone segments $\mathrm{Clo}_i(A_1)$ and $\mathrm{Clo}_i(A_2)$ of finitary algebras A_1 and A_2 which, as indicated earlier, are the algebraic traces of suitable disciplined metrizable spaces. The equality of centralizer clone segments of such algebras has already been treated by Proposition 2.9. Here we investigate their isomorphism, for which we prove a similar Proposition 3.5 and, in 3.8, also set down a basis for subsequent consideration of elementary equivalence.

We need the following concepts.

3.2. Definition. (a) Let $F: k \longrightarrow \mathbf{Set}$ be a functor and let a be an object of the category k. We say that an element $x \in F(a)$ is an *absolute fixpoint* of a whenever, for any morphism $f \in k(a,a)$, either the mapping $F(f)$ is constant or $F(f)(x) = x$.

(b) Let $a \in \operatorname{obj} k$ generate a clone (k, F), and let $n \in \omega$. A morphism $c \in k(a^n, a)$ is called an *absolute constant* whenever $F(c): F(a^n) \longrightarrow F(a)$ is a constant map whose value is an absolute fixpoint of a.

3.3 . We recall that the first order language of clones is just the first order language of ω-sorted algebras with the operations $c_j^{(n)}$ and S_m^n with $j \in n$ and $m, n \in \omega$, as described in 1.8. The language has ω sorts of variables, with symbols $x^{(i)}, y^{(i)}, x_1^{(i)}, x_2^{(i)}, \dots$ denoting the variables of the i-th sort for each $i \in \omega$. For any $r \geq 1$, the first order language of clone r-segments is the fragment of the first order language of clones which has variables $x^{(i)}$, $y^{(i)}, \dots$ with $i = 0, 1, \dots, r$ only, and operation symbols S_m^n and $c_j^{(n)}$ with $j \in n$ restricted to $m, n \leq r$.

Since in our case the variables of the i-th sort range through maps $Y^i \longrightarrow Y$, we simplify matters somewhat by treating all variables as maps. As is customary, we shall identify elements of Y with constant maps $Y \longrightarrow Y$, so that a variable $x^{(0)}$ will be replaced by a variable $x^{(1)}$ of the 1-st sort satisfying the predicate

$$\mathrm{p}(x^{(1)}) \equiv \forall y^{(1)} S_1^1(x^{(1)}, y^{(1)}) = x^{(1)}.$$

In what follows, we shall also use the predicates

$$\mathrm{afp}(x^{(1)}) \equiv \mathrm{p}(x^{(1)}) \wedge \forall y^{(1)}\big(\mathrm{p}(y^{(1)}) \vee S_1^1(y^{(1)}, x^{(1)}) = x^{(1)}\big) \text{ and}$$
$$\mathrm{ac}(z^{(n)}) \equiv \exists x^{(1)}\big(\mathrm{afp}(x^{(1)}) \wedge S_n^1(x^{(1)}, z^{(n)}) = z^{(n)}\big),$$

which describe absolute fixpoints and absolute constants in this language.

3.4 . If $X = (P, t)$ is an infinite disciplined metrizable space then, by Lemma 2.5, any homomorphism $h: \mathbb{A}(X) \longrightarrow \mathbb{A}(X)$ with $h(\lambda) \neq \lambda$ is a constant map (whose value belongs to P). Whence λ is an absolute fixpoint of $\mathbb{A}(X)$, and the homomorphism $\lambda^{(n)}: \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ is an absolute constant in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ for every $n \geq 0$.

If X has no absolute fixpoints, then for every $p \in P$ there is a non-constant continuous map $f_p: X \longrightarrow X$ with $f_p(p) \neq p$. Hence the unique homomorphism $E(f_p): \mathbb{A}(X) \longrightarrow \mathbb{A}(X)$ extending f_p—see Proposition 2.3(1)—is non-constant and such that $E(f_p)(s) \neq s$ for any $s \in S$ with $\pi_0(s) = p$. It follows that $\mathbb{A}(X)$ has no absolute fixpoints other than λ, that $\lambda^{(n)}$ is the only absolute constant $\mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$, and that $\mathrm{Clo}(X)$ has no absolute constants.

Hence, by 2.3, for every $n \in \omega$,

$$\mathrm{Clo}\big(\mathbb{A}(X)\big)_n = \{E(f) \mid f \in \mathrm{Clo}(X)_n\} \cup \{\lambda^{(n)}\},$$

and $\lambda^{(n)}$ is the unique absolute constant in $\mathrm{Clo}\big(\mathbb{A}(X)\big)_n$. Let

$$E: \mathrm{Clo}(X) \longrightarrow \mathrm{Clo}\big(\mathbb{A}(X)\big)$$

denote both the embedding of these clone algebras and also the functor preserving finite products and projection enumeration corresponding to it.

3.5. PROPOSITION. *Let $k \geq 1$ be an integer, and let X_1 and X_2 be infinite disciplined metrizable spaces without absolute fixpoints. Then*

$$\mathrm{Clo}_k(X_1) \simeq \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \simeq \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

This equivalence also holds for the respective full clones.

PROOF. In the diagram below, let $E^{(1)}$ and $E^{(2)}$ be restrictions of embeddings from 3.4 above to the respective clone segments.

$$\begin{tikzcd}
\mathrm{Clo}_k(X_1) \arrow[r, "E^{(1)}"] \arrow[d, "\Phi"'] & \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \arrow[d, "\Psi"] \\
\mathrm{Clo}_k(X_2) \arrow[r, "E^{(2)}"] & \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)
\end{tikzcd}$$

Suppose first that the functor Ψ in this diagram is an isomorphism. Then Ψ preserves elements satisfying the same first order formulas, and hence it must map absolute constants of its domain precisely onto absolute constants in its codomain. By 3.4, there is a unique isomorphism Φ completing the diagram above.

Conversely, let Φ be an isomorphism. We extend Φ to a product-preserving functor Ψ which sends the unique absolute constant $\lambda^{(n)}$ of $\mathrm{Clo}\big(\mathbb{A}(X_1)\big)_n$ to the unique absolute constant of $\mathrm{Clo}\big(\mathbb{A}(X_2)\big)_n$ for every $n \leq k$. Then the diagram above commutes. The fact that the bijection Ψ is a clone isomorphism or, equivalently, that it preserves all operations S^n_m with $m, n \leq k$, will follow from Lemma 3.8 below. The remainder is clear. ∎

3.6 . Let $K = (k, F)$ be a clone with the generating object a, and let $\psi: r \longrightarrow n$ be a map for some $r, n \in \omega$. As in 1.2 for clones of spaces, but now for any clone, we write $\pi^{[\psi]}: a^n \longrightarrow a^r$ to denote the k-morphism

$$\pi^{[\psi]} = \pi^{(n)}_{\psi(0)} \dot{\times} \cdots \dot{\times} \pi^{(n)}_{\psi(r-1)}.$$

Then, for any $h \in k(a^r, a)$,

$$h \circ \pi^{[\psi]} = S^r_n(h, \pi^{(n)}_{\psi(0)}, \ldots, \pi^{(n)}_{\psi(r-1)})$$

follows from the definition of clone algebra operations.

3.7 . Let $K = (k, F)$ be a clone with the generating object a. Since F is faithful and $a^n \neq a^m$ for distinct $n, m \in \omega$, any k-morphism $g \in k(a^n, a)$ is fully determined by $F(g)$. Hence there is no need to distinguish $F(g)$ from g, and we shall write g instead of $F(g)$ as well.

DEFINITION. Let $g \in k(a^n, a)$. We say that

$$g = h \circ \pi^{[\psi]}$$

is an *extremal decomposition* of g—in the clone K or in any of its m-segments with $m \geq n$—whenever $\psi: r \longrightarrow n$ is a one-to-one map, and for any decomposition $g = h_1 \circ \pi^{[\psi_1]}$

with a one-to-one map $\psi_1 : s \longrightarrow n$ there exists a k-morphism $d \in k(a^s, a^r)$ such that $d \circ \pi^{[\psi_1]} = \pi^{[\psi]}$.

REMARKS. (a) Since ψ and ψ_1 in the definition above are both one-to-one, we have $r, s \leq n$. Thus the definition does apply not only to the full clone, but also to all its m-segments with $m \geq n$.

(b) Any factorizing morphism d is unique and has the form $d = \pi^{[\delta]}$ for some δ with $\psi_1 \circ \delta = \psi$.

(c) Since the map $\pi^{[\psi]}$ is surjective for any injective map ψ, an extremal decomposition is determined uniquely up to a commuting isomorphism.

OBSERVATION. If $g = h \circ \pi^{[\psi]} \in k(a^n, a)$ with $\psi : r \longrightarrow n$, and if $f_j \in k(a^m, a)$ for $j \in n$, then

$$\begin{aligned} S^n_m(g, f_0, \dots, f_{n-1}) &= g \circ (f_0 \dot\times \cdots \dot\times f_{n-1}) = h \circ (f_{\psi(0)} \dot\times \cdots \dot\times f_{\psi(r-1)}) \\ &= S^r_m(h, f_{\psi(0)}, \dots, f_{\psi(r-1)}). \end{aligned}$$

This follows immediately from $\pi^{[\psi]} = \pi^{(n)}_{\psi(0)} \dot\times \cdots \dot\times \pi^{(n)}_{\psi(r-1)}$.

3.8 . The Lemma below summarizes the relationship of $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ to $\mathrm{Clo}(X)$ in a form suitable for investigations of elementary equivalence.

Let $E : \mathrm{Clo}(X) \longrightarrow \mathrm{Clo}\big(\mathbb{A}(X)\big)$ be the embedding from 3.4. Proposition 2.3 shows that, for each $i \in n$, the homomorphism $E(\pi^{(n)}_i) \in \mathrm{Clo}\big(\mathbb{A}(X)\big)_n$ is the i-th projection, and that $E(\pi^{(n)}_i)$ uniquely extends the i-th projection $\pi^{(n)}_i \in \mathrm{Clo}(X)_n$. It follows that $E(\pi^{[\psi]})$ is the unique extension of $\pi^{[\psi]} = \pi^{(n)}_{\psi(0)} \dot\times \cdots \dot\times \pi^{(n)}_{\psi(r-1)}$ for any $\psi : r \longrightarrow n$ as well. This allows us to simplify the notation as follows: we write $\pi^{(n)}_i$ instead of $E(\pi^{(n)}_i) \in \mathrm{Clo}\big(\mathbb{A}(X)\big)$ and, more generally, $\pi^{[\psi]}$ instead of $E(\pi^{[\psi]}) \in \mathrm{Clo}\big(\mathbb{A}(X)\big)$.

LEMMA. *If $X = (P, t)$ is a disciplined metrizable space and $n \in \omega$, then* $\mathrm{Clo}(X)$ *and* $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ *have the following properties:*

(1) $\mathrm{Clo}\big(\mathbb{A}(X)\big)_n = \{E(g') \mid g' \in \mathrm{Clo}(X)_n\} \cup \{\lambda^{(n)}\}$.

(2) For any $g' \in \mathrm{Clo}(X)_n$, any standard decomposition $g' = h' \circ \pi^{[\psi]}$ of g' is also an extremal decomposition of g'.

(3) Every $g = E(g') \in \mathrm{Clo}\big(\mathbb{A}(X)\big)_n \setminus \{\lambda^{(n)}\}$ has an extremal decomposition $g = E(h') \circ \pi^{[\psi]}$, where $g' = h' \circ \pi^{[\psi]}$ is an extremal decomposition of g'.

(4) Let $f = (f_0 \dot\times \cdots \dot\times f_{n-1}) : \mathbb{A}(X)^m \longrightarrow \mathbb{A}(X)^n$ and $g : \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$ be in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$. *Then either $g = \lambda^{(n)}$ and hence $g \circ f = \lambda^{(m)}$, or else g has an extremal decomposition $g = E(h') \circ \pi^{[\psi]}$ and*

$$g \circ f = \begin{cases} \lambda^{(m)} & \text{if } f_{\psi(i)} = \lambda^{(m)} \text{ for some } i \in r \\ E(h') \circ (f_{\psi(0)} \dot\times \cdots \dot\times f_{\psi(r-1)}) & \text{otherwise,} \end{cases}$$

with $f_{\psi(i)} = E(f'_{\psi(i)})$ for all $i \in r$.

These conclusions also hold for any initial k-segments of these clones when $k \geq 1$.

PROOF. Claim (1) is already contained in 3.4. To prove (2), let $g' = h' \circ \pi^{[\psi]}$ be a standard decomposition. To show that this is also an extremal decomposition, suppose that $g' = h_1 \circ \pi^{[\psi_1]}$ for some injective $\psi_1 : s \longrightarrow n$. Since the projections $\pi^{[\psi]}$ and $\pi^{[\psi_1]}$ are surjective and h' is a homeomorphism of X^r onto $h_1(X^s)$, the composite $d = (h')^{-1} \circ h_1$ exists and is continuous. But then $h' \circ d \circ \pi^{[\psi_1]} = h' \circ \pi^{[\psi]}$, and $d\pi^{[\psi_1]} = \pi^{[\psi]}$ follows because h' is injective.

Now we turn to (3). If $g \neq \lambda^{(n)}$, then $g = E(g')$ for a unique $g' : X^n \longrightarrow X$, by (1), and g' has an extremal decomposition $g' = h' \circ \pi^{[\psi]}$, by (2). From Lemma 2.6 it then follows that $E(\pi^{[\psi]}) = \pi^{[\psi]}$ and $g = E(h') \circ \pi^{[\psi]}$. To show that this is an extremal decomposition of g, suppose that $g = h_1 \circ \pi^{[\psi_1]}$ with an injective $\psi_1 : s \longrightarrow n$. Then $h_1 \neq \lambda^{(s)}$ because $g \neq \lambda^{(n)}$, so that $h_1 = E(h_1')$ for some $h_1' \in \mathrm{Clo}(X)_s$, by (1) again, and $g' = h_1' \circ \pi^{[\psi_1]}$ follows. Since the decomposition $g' = h' \circ \pi^{[\psi]}$ is extremal, we have $d \circ \pi^{[\psi_1]} = \pi^{[\psi]}$ for some $d \in \mathrm{Clo}(X)$, and hence $E(d) \circ \pi^{[\psi_1]} = \pi^{[\psi]}$ in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$. Therefore $g = E(h') \circ \pi^{[\psi]}$ is an extremal decomposition of g, as claimed.

To prove (4), let $g \neq \lambda^{(n)}$. Then $g = E(g') = E(h') \circ \pi^{[\psi]}$ is an extremal decomposition of g in which h' is a homeomorphism of X^r onto a closed subset of X, by (3) and (2), and $g \circ f = E(h') \circ (f_{\psi(0)} \dot{\times} \cdots \dot{\times} f_{\psi(r-1)})$ by Observation 3.7. From the definition of the extension $E(h')$ of the homeomorphism h' it then follows that $g \circ f = \lambda^{(m)}$ whenever $f_{\psi(i)} = \lambda^{(m)}$ for at least one $i \in r$. If, on the other hand, $f_{\psi(i)} \neq \lambda^{(m)}$ for every $i \in r$, then $f_{\psi(i)} = E(f'_{\psi(i)})$ for all $i \in r$ by (1), and (4) follows. ■

4. **Elementary equivalence.**

4.1. The purpose of this section is to verify the following claim about elementary equivalence of clones or their segments.

PROPOSITION. *For any two infinite disciplined metrizable spaces X_1 and X_2 with no absolute fixpoints, and for any $k \geq 1$,*

$$\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

This equivalence also holds for the respective full clones.

Even though it is easy to show, using Lemma 3.8 and a natural uniform interpretation of sentences from the theory of $\mathrm{Clo}_k(X)$ in the theory of $\mathrm{Clo}_k\big(\mathbb{A}(X)\big)$, that $\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2)$ follows from $\mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$, sentences claiming the existence of absolute constants make a converse interpretation unwieldy, and we employ other means to prove the above Proposition. Specifically, we use the well-known fact that elementary equivalence is equivalent to isomorphism of prime limits (see [4] for monosorted algebras). We prove that an isomorphism of certain prime limits of $\mathrm{Clo}_k(X_1)$ and $\mathrm{Clo}_k(X_2)$ can be extended to an isomorphism of associated prime limits of $\mathrm{Clo}_k\big(\mathbb{A}(X_1)\big)$ and $\mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$ and, vice versa, that a restriction of an isomorphism of some prime limits of $\mathrm{Clo}_k\big(\mathbb{A}(X_1)\big)$ and $\mathrm{Clo}_k\big(\mathbb{A}(X_2)\big)$ is an isomorphism of the associated prime limits of $\mathrm{Clo}_k(X_1)$ and $\mathrm{Clo}_k(X_2)$.

4.2. In order to introduce an appropriate notation, we recall how ultrapowers of clone algebras are created.

Let K be a clone algebra as described in 1.8, and let K_n denote its carrier of the n-th sort. Given an ultrafilter $\mathcal{F}$ on a set I, for any $s, t \in K_n^I$ we define

$$s \sim_{\mathcal{F}} t \text{ iff } \mathrm{Eq}(s,t) = \{i \in I \mid s(i) = t(i)\} \in \mathcal{F}.$$

Then $\sim_{\mathcal{F}}$ is an equivalence. For any $s \in K_n^I$, the symbol $[s]_{\mathcal{F}}$ will denote the class of $\sim_{\mathcal{F}}$ containing s. If $\kappa_n^{(\mathcal{F})}$ is the natural map of K_n^I onto the ultrapower $K_n^{(\mathcal{F})} = K_n^I / \sim_{\mathcal{F}}$ with the kernel $\sim_{\mathcal{F}}$ then $\kappa_n^{(\mathcal{F})}(s) = [s]$, so that each $\sigma \in K_n^{(\mathcal{F})}$ may be replaced, as is customary, by any $s \in K_n^I$ with $[s]_{\mathcal{F}} = \sigma$. Whence $t \in [s]_{\mathcal{F}}$ exactly when $\mathrm{Eq}(s,t) \in \mathcal{F}$.

For any $w \in K_n$, we let $\bar{w} \in K_n^I$ stand for the constant sequence whose value is w. For each $j \in n$ we then write

$$(c_j^{(n)})^{(\mathcal{F})} = [\overline{c_j^{(n)}}]_{\mathcal{F}} \in K_n^{(\mathcal{F})},$$

where $c_j^{(n)}$ is the nullary operation from 1.8(c).

Let $(S_m^n)^I : K_n^I \times (K_m^I)^n \longrightarrow K_m^I$ be the componentwise extension of an operation $S_m^n : K_n \times K_m^n \longrightarrow K_m$ from 1.8(s), that is, let

$$\big((S_m^n)^I(s, t_0, \dots, t_{n-1})\big)(i) = S_m^n\big(s(i), t_0(i), \dots t_{n-1}(i)\big)$$

for every $i \in I$, any $s \in K_n$ and any $t_0, \dots, t_{n-1} \in K_m$. If $s' \in [s]_{\mathcal{F}}$ and $t_j' \in [t_j]_{\mathcal{F}}$ for all $j \in n$, then

$$\mathrm{Eq}(s, s') \cap \bigcap \{\mathrm{Eq}(t_j, t_j') \mid j \in n\} \in \mathcal{F},$$

and hence also

$$\mathrm{Eq}\big((S_m^n)^I(s, t_0, \dots, t_{n-1}), (S_m^n)^I(s', t_0', \dots, t_{n-1}')\big) \in \mathcal{F}.$$

It follows that there is a well-defined and unique mapping

$$(S_m^n)^{(\mathcal{F})} : K_n^{(\mathcal{F})} \times (K_m^{(\mathcal{F})})^n \longrightarrow K_m^{(\mathcal{F})}$$

given for $\sigma \in K_n^{(\mathcal{F})}$ and $\tau_0, \dots \tau_{n-1} \in K_m^{(\mathcal{F})}$ by

$$(S_m^n)^{(\mathcal{F})}(\sigma, \tau_0, \dots, \tau_{n-1}) = [(S_m^n)^I(s, t_0, \dots, t_{n-1})]_{\mathcal{F}}$$

with an arbitrary choice of $s \in \sigma$ and $t_j \in \tau_j$ for each $j \in n$.

Let $\chi : K \longrightarrow K^{(\mathcal{F})}$ be the diagonal map given by $\chi(k) = [\bar{k}]_{\mathcal{F}} \in K_n^{(\mathcal{F})}$ for all $k \in K_n$. Then χ is a one-to-one homomorphism of the clone algebra K into a heterogeneous algebra $K^{(\mathcal{F})}$ whose operations $(c_j^{(n)})^{(\mathcal{F})}$ and $(S_m^n)^{(\mathcal{F})}$ satisfy (RU), (LU) and (AC) from 1.8. Having identified every $k \in K$ with its image $\chi(k) \in K^{(\mathcal{F})}$, we thus obtain a clone algebra $K^{(\mathcal{F})}$ whose carrier of the n-th sort is $K_n^{(\mathcal{F})}$ and whose operations are $(c_j^{(n)})^{(\mathcal{F})} = c_j^{(n)}$ and $(S_m^n)^{(\mathcal{F})}$. The clone algebra $K^{(\mathcal{F})}$ will be called a *prime power* of K associated with $\mathcal{F}$.

Let K, L be clone algebras, and let $E : K \longrightarrow L$ be a clone homomorphism. The definition of its extension $E^{(\mathcal{F})} : K^{(\mathcal{F})} \longrightarrow L^{(\mathcal{F})}$ is evident, and it is clear that $E^{(\mathcal{F})}$ is an embedding whenever E is an embedding.

4.3 . For any integer $r \geq 1$, let $\mathcal{F}_r$ be an ultrafilter on some set I_r and let $\mathbb{F} = \{\mathcal{F}_r \mid r \geq 1\}$. Set $K_{(0)} = K$, then inductively define $K_{(r)} = K_{(r-1)}^{(\mathcal{F}_r)}$ for each $r \geq 1$, and let $\chi_r : K_{(r-1)} \longrightarrow K_{(r)}$ denote the diagonal embedding arising at each step. If $0 \leq s < r$, then $\chi_r \circ \cdots \circ \chi_{s+1}$ is an injective homomorphism, and we may therefore assume that $K_{(s)}$ is a subalgebra of $K_{(r)}$ whenever $0 \leq s < r$. These composite embeddings, or rather inclusions, form a diagram whose colimit $\mathrm{PL}_{\mathbb{F}}(K)$ is an ω-sorted algebra isomorphic to the union of the ω-chain

$$K_{(0)} \xrightarrow{\chi_1} K_{(1)} \xrightarrow{\chi_2} K_{(2)} \longrightarrow \quad \cdots \quad ,$$

with colimit injections $\mu_s : K_{(s)} \longrightarrow \mathrm{PL}_{\mathbb{F}}(K)$. It is clear that the ω-sorted algebra $\mathrm{PL}_{\mathbb{F}}(K)$, called a *prime limit* of K over $\mathbb{F}$, satisfies (LU), (RU) and (AC) from 1.8 with the nullary operations $c_j^{(n)} \in K_{(0)}$ for all $j \in n$ and joint extensions $(S_m^n)^{(\mathbb{F})}$ of all $(S_m^n)^{(\mathcal{F}_r)}$ with $r \geq 1$.

For a clone homomorphism $E : K \longrightarrow L$, the definition of $\mathrm{PL}_{\mathbb{F}}(E) : \mathrm{PL}_{\mathbb{F}}(K) \longrightarrow \mathrm{PL}_{\mathbb{F}}(L)$ is evident. It is clear that if E is an embedding then $\mathrm{PL}_{\mathbb{F}}(E)$ is also an embedding.

4.4 . Let X be an infinite disciplined space with no absolute fixpoints, and let $\mathbb{F}$ be a sequence of ultrafilters. In the lemma below, we extend claims (1)–(4) of Lemma 3.8 about the clone algebras $\mathrm{Clo}(X)$ and $\mathrm{Clo}\big(\mathbb{A}(X)\big)$ to analogous claims about their prime limits $A = \mathrm{PL}_{\mathbb{F}}\big(\mathrm{Clo}(X)\big)$ and $A^* = \mathrm{PL}_{\mathbb{F}}\Big(\mathrm{Clo}\big(\mathbb{A}(X)\big)\Big)$.

The non-nullary clone operations will be denoted as S_n^m in A, and as $(S_n^m)^*$ in A^*, while $c_i^{(n)}$ will denote the clone constant corresponding to the product projection $\pi_i^{(n)}$ in $\mathrm{Clo}(X)$ and in $\mathrm{Clo}\big(\mathbb{A}(X)\big)$, in both A and A^*.

From 3.7 we recall that any existing extremal decomposition is determined uniquely up to a commuting isomorphism. We note that, in the lemma below, all decompositions are expressed by means of clone algebra operations, see Observation 3.7.

LEMMA. *Let X, $\mathbb{F}$, A, A^* be as above. Then:*

(1) *There exists a one-to-one clone homomorphism $E : A \longrightarrow A^*$ such that the carrier A_n^* of the n-th sort of A^* is the disjoint union $A_n^* = \{E(g') \mid g' \in A_n\} \cup \{*^{(n)}\}$, and $*^{(n)}$ is the only absolute constant in A_n^*.*

(2) *Each $g' \in A_n$ has an extremal decomposition $g' = S_n^r(h', c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$.*

(3) *$E : A \longrightarrow A^*$ preserves extremal decompositions, that is, if $g \in A_n^*$, $g = E(g')$ and $g' = S_n^r(h', c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$ is an extremal decomposition of g' in A, then $g = (S_n^r)^*\big(E(h'), c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)}\big)$ is an extremal decomposition of g in A^*.*

(4) *For $f_0, \ldots, f_{n-1} \in A_m^*$ and $g \in A_n^*$, denote*

$$C = (S_m^n)^*(g, f_0, \ldots, f_{n-1}).$$

Then $C = *^{(m)}$ for $g = *^{(n)}$. Otherwise $g \in A_n^* \setminus \{*^{(n)}\}$ has an extremal decomposition $g = (S_n^r)^*(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$, by (1)–(3), and

$$C = \begin{cases} *^{(m)} & \text{if } f_{\psi(i)} = *^{(m)} \text{ for some } i \in r \\ (S_m^r)^*(h, f_{\psi(0)}, \ldots, f_{\psi(r-1)}) & \text{if } f_{\psi(i)} \neq *^{(m)} \text{ for all } i \in r. \end{cases}$$

The same conclusions hold for prime limits $\mathrm{PL}_{\mathbb{F}}\bigl(\mathrm{Clo}_k(X)\bigr)$ *and* $\mathrm{PL}_{\mathbb{F}}\Bigl(\mathrm{Clo}_k\bigl(\mathbb{A}(X)\bigr)\Bigr)$ *of k-segments of the clone algebras.*

PROOF. Denote $A_{(0)} = \mathrm{Clo}(X)$, $A^*_{(0)} = \mathrm{Clo}\bigl(\mathbb{A}(X)\bigr)$, and let $E_{(0)}: A_{(0)} \longrightarrow A^*_{(0)}$ be as in 3.4. For every $s \geq 1$, write $A_{(s)} = (A_{(s-1)})^{(\mathcal{F}_s)}$, $A^*_{(s)} = (A^*_{(s-1)})^{(\mathcal{F}_s)}$ and $E_{(s)} = (E_{(s-1)})^{(\mathcal{F}_s)}$, where $\mathcal{F}_s \in \mathbb{F}$ is an ultrafilter on a set I_s.

From 3.4 and Lemma 3.8 it follows that $A_{(0)}, A^*_{(0)}$ and $E_{(0)}$ already satisfy (1), (2), (3) and (4). From the definition of a prime limit it follows that the lemma will be proved once we show that these four statements hold for $A_{(s)}, A^*_{(s)}$ and $E_{(s)}$ whenever they hold for $A_{(s-1)}, A^*_{(s-1)}$ and $E_{(s-1)}$.

Hence, let an ultrafilter $\mathcal{F}$ on a set I be given. We shall suppose that (1)–(4) hold for clone algebras $K, L = K^*$ and a clone-embedding $E: K \longrightarrow L$, and prove these statements for the ultrapowers $K^{(\mathcal{F})}$, $L^{(\mathcal{F})}$, and $E^{(\mathcal{F})}$. To simplify the notation, let us suppose that the embedding E is an inclusion, so that K is a subalgebra of L and each operation $(S^n_m)^*$ of L is a mere extension of the operation S^n_m of K.

To prove (1), assume that the carrier L_n of the n-th sort of L is the disjoint union $L_n = K_n \cup \{*^{(n)}\}$, in which $*^{(n)}$ is the only absolute constant. Clearly $K^{(\mathcal{F})}_n \subseteq L^{(\mathcal{F})}_n$. Furthermore, the sentence $\forall x^{(n)} \neg\, \mathrm{ac}(x^{(n)})$ holds in K_n, while its negation $\exists x^{(n)}\, \mathrm{ac}(x^{(n)})$ is satisfied in L_n by $*^{(n)}$ alone. For an arbitrary $f \in L^{(\mathcal{F})}_n$, the set I is the disjoint union of $J = \{i \in I \mid f(i) \in K_n\}$ and $J' = \{i \in I \mid f(i) = *^{(n)}\}$, so that exactly one of these sets belongs to $\mathcal{F}$. If $J \in \mathcal{F}$, then $f \in K^{(\mathcal{F})}_n$ and f is not an absolute constant. Else $J' \in \mathcal{F}$ and $f \sim_{\mathcal{F}} \overline{*^{(n)}}$, and f must be the absolute constant in $L^{(\mathcal{F})}_n$. Having identified diagonal members of $L^{(\mathcal{F})}_n$ with their values in L_n, we obtain (1) for $K^{(\mathcal{F})}_n$ and $L^{(\mathcal{F})}_n$.

For (2), select any $g \in K^{(\mathcal{F})}_n$. Then $J = \{i \in I \mid g(i) \in K_n\} \in \mathcal{F}$. Since $g(i) \neq *^{(n)}$ for all $i \in J$, every such $g(i)$ has an extremal decomposition $g(i) = S^{r_i}_n\bigl(h(i), c^{(n)}_{\psi_i(0)}, \ldots, c^{(n)}_{\psi_i(r_i-1)}\bigr)$ in K_n. Each $\psi_i: r_i \longrightarrow n$ is injective, there are only finitely many of these maps, and hence there is a unique injective map $\psi: r \longrightarrow n$ for which $J_\psi = \{i \in J \mid \psi_i = \psi\} \in \mathcal{F}$. But then $g = S^r_n(h, c^{(n)}_{\psi(0)}, \ldots, c^{(n)}_{\psi(r-1)})$ in $K^{(\mathcal{F})}$ because $h(i) \in K_n$ for every $i \in J$.

To see that this decomposition is extremal, let also $g = S^{r_1}_n(h_1, c^{(n)}_{\psi_1(0)}, \ldots, c^{(n)}_{\psi_1(r_1-1)})$ with an injective map $\psi_1: r_1 \longrightarrow n$. Then there exists some $J_1 \in \mathcal{F}$ so that the left hand side in

$$S^r_n\bigl(h(i), c^{(n)}_{\psi(0)}, \ldots, c^{(n)}_{\psi(r-1)}\bigr) = S^{r_1}_n\bigl(h_1(i), c^{(n)}_{\psi_1(0)}, \ldots, c^{(n)}_{\psi_1(r_1-1)}\bigr)$$

is an extremal decomposition of $g(i)$ in K, and the equality itself holds for every $i \in J_1$. Since $J_1 \neq \emptyset$, there exists a unique $\delta: r \longrightarrow r_1$ with $\psi_1 \circ \delta = \psi$ and, because $J_1 \in \mathcal{F}$, the decomposition $g = S^r_n(h, c^{(n)}_{\psi(0)}, \ldots, c^{(n)}_{\psi(r-1)})$ of g is extremal. This proves (2).

For (3), we need that an extremal decomposition $g = S^r_n(h, c^{(n)}_{\psi(0)}, \ldots, c^{(n)}_{\psi(r-1)})$ in $K^{(\mathcal{F})}_n$ of any $g \in K^{(\mathcal{F})}_n$ is its extremal decomposition also in $L^{(\mathcal{F})}_n$. This follows easily from the fact that (4) holds for K and L.

To prove (4), let $f_0, \ldots, f_{n-1} \in L^{(\mathcal{F})}_m$, $g \in L^{(\mathcal{F})}_n$, and let $C = (S^n_m)^*(g, f_0, \ldots, f_{n-1})$ be the composite in $L^{(\mathcal{F})}$. If $g = *^{(n)}$, then $C = *^{(m)}$ because $*^{(n)}$ is a constant. Any

$g \in L_n^{(\mathcal{F})} \setminus \{*^{(n)}\}$ belongs to $K_n^{(\mathcal{F})}$, by (1), and hence has an extremal decomposition $g = S_n^r(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$, by (2). If $f_{\psi(j)} = *^{(m)}$ for some $j \in r$, then $C = *^{(n)}$ follows from the definition of $(S_m^n)^*$ on the ultrapower $L^{(\mathcal{F})}$ and the fact that $*^{(m)}$ is constant. Otherwise also $f_{\psi(j)} \in K_m^{(\mathcal{F})}$ for all $j \in n$, by (1), and hence there exists some $J_2 \in \mathcal{F}$ such that $C(i) = S_m^n\big(g(i), f_0(i), \ldots, f_{n-1}(i)\big)$ in K for all $i \in J_2$. But then $C = S_m^n(g, f_0, \ldots, f_{n-1})$ in $K^{(\mathcal{F})}$, and this completes the proof of (4) for $K^{(\mathcal{F})}$ and $L^{(\mathcal{F})}$. ■

4.5. LEMMA. *Let $E_A: A \longrightarrow A^*$ and $E_B: B \longrightarrow B^*$ be clone algebra embeddings satisfying (1)–(4) of Lemma 4.4. Then*

$$A \simeq B \text{ if and only if } A^* \simeq B^*.$$

PROOF. To simplify the notation, let us suppose again that the embeddings E_A, E_B are inclusions. We let $(S_m^n)_A, (S_m^n)_A^*, (S_m^n)_B, (S_m^n)_B^*$ stand for the non-nullary clone operations of A, A^*, B, B^* respectively, and $c_i^{(n)}$ for the nullary operations of each of these four algebras.

Let $\Phi: A \longrightarrow B$ be a clone algebra isomorphism. Then Φ maps each sort A_n of A bijectively onto the sort B_n of B in such a way that $\Phi(c_j^{(n)}) = c_j^{(n)}$ for every $j \in n$. Furthermore, if $g = (S_n^r)_A(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$ is an extremal decomposition of some $g \in A_n$ then $\Phi(g) = (S_n^r)_B\big(\Phi(h), c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)}\big)$ is an extremal decomposition of $\Phi(g)$.

From (1) we recall that $A_n^* = A_n \cup \{*^{(n)}\}$ and $B_n^* = B_n \cup \{*^{(n)}\}$, and set

$$\Psi(g) = \begin{cases} \Phi(g) & \text{for all } g \in A_n \\ *^{(n)} & \text{for } g = *^{(n)}. \end{cases}$$

It is clear that Ψ maps each A_n^* bijectively onto B_n^*. To prove that Ψ is an isomorphism, we only need to show that

$$\text{(C)} \qquad \Psi\big((S_m^n)_A^*(g, f_0, \ldots, f_{n-1})\big) = (S_m^n)_B^*\big(\Psi(g), \Psi(f_0), \ldots, \Psi(f_{n-1})\big)$$

for any choice of $f_0, \ldots, f_{n-1} \in A_m^*$ and $g \in A_n^*$.

First suppose that $g = *^{(n)}$.

Here $(S_m^n)_A^*(g, f_0, \ldots, f_{n-1}) = *^{(m)}$ and $(S_m^n)_B^*\big(\Psi(g), \Psi(f_0), \ldots, \Psi(f_{n-1})\big) = *^{(m)}$, from (4) applied to both A^* and B^*. But then (C) follows because $\Psi(*^{(m)}) = *^{(m)}$.

Let $g \neq *^{(n)}$ next.

Then g has an extremal decomposition $g = (S_n^r)_A(h, c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)})$ with $h \neq *^{(r)}$, and hence $\Psi(g) = (S_n^r)_B\big(\Phi(h), c_{\psi(0)}^{(n)}, \ldots, c_{\psi(r-1)}^{(n)}\big)$ is an extremal decomposition of $\Psi(g)$. If $f_{\psi(i)} = *^{(m)}$ for some $i \in r$, then $\Psi(f_{\psi(i)}) = *^{(m)}$, and (C) follows from an application of (4) to both A^* and B^*. Otherwise $f_{\psi(i)} \in A_n$ for all $i \in r$ by (1), and (C) obtains from (4) because Φ is an isomorphism of A onto B. Therefore $\Psi: A^* \longrightarrow B^*$ is an isomorphism, as claimed.

Conversely, assume $\Psi: A^* \longrightarrow B^*$ to be an isomorphism of these clone algebras. If $f \in A_n$ then $\neg\mathrm{ac}(f)$ by (1), and hence $\neg\mathrm{ac}\big(\Psi(f)\big)$ because $\neg\mathrm{ac}(x^{(n)})$ is a first order formula and Ψ is an isomorphism. But then $\Psi(f) \in B_n$ by (1) again and, therefore, the restriction Φ of Ψ to A maps A isomorphically onto B. ■

4.6. Now we are ready to complete the proof of Proposition in 4.1, which claims that for any two infinite disciplined metrizable spaces X_1, X_2 with no absolute fixpoints and any $k \in \{1 \ldots, \infty\}$,

$$\mathrm{Clo}_k(X_1) \approx \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k\big(\mathbb{A}(X_1)\big) \approx \mathrm{Clo}_k\big(\mathbb{A}(X_2)\big).$$

But this follows immediately from 4.4–5 and the fact that, similarly to monosorted algebras (see [4] for instance), heterogeneous algebras of the same type are elementarily equivalent if and only if they have isomorphic prime limits.

4.7. In 1.9, we noted the following result.

THEOREM [7]. *If $m, n, p \in \{1, 2, \ldots, \omega\}$ satisfy $m \leq n \leq p$, then there exist infinite metrizable spaces X_1 and X_2 such that*

$$\begin{aligned} m &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) = \mathrm{Clo}_i(X_2)\}, \\ n &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \simeq \mathrm{Clo}_i(X_2)\}, \\ p &= \sup\{i \leq \omega \mid \mathrm{Clo}_i(X_1) \approx \mathrm{Clo}_i(X_2)\}. \end{aligned}$$ ■

Thus to demonstrate the claim of Theorem 1.9 about the three clone segments, in view of Propositions 2.9, 3.5 and 4.1 we need only show that the spaces X_1 and X_2 of Theorem [7] may be chosen to be disciplined and without absolute fixpoints. This we do in the next section.

5. Disciplined spaces with no absolute fixpoints abound.

5.1. Let $\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$ be a finitary type of (mono-sorted) universal algebras in which Σ_n denotes the set of all n-ary operation symbols. For $\sigma \in \Sigma$, we write $\mathrm{ar}\,\sigma = n$ whenever $\sigma \in \Sigma_n$, and always assume that Σ_0 is infinite.

Let $\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$ be an absolutely free Σ-algebra over the empty set of generators. It is well-known that $P = \bigcup_{k=0}^{\infty} P_k$, with

$$P_0 = \{p_\sigma \mid \sigma \in \Sigma_0\} \text{ and } P_{k+1} = P_k \cup \bigcup \{p_\sigma(P_k^{\mathrm{ar}\,\sigma}) \mid \sigma \in \Sigma \setminus \Sigma_0\},$$

where the unions are disjoint and every p_σ with $\sigma \in \Sigma \setminus \Sigma_0$ is one-to-one.

5.2. For any integer $m \geq 1$, let $\mathbb{T}(x_0, \ldots, x_{m-1})$ denote the term algebra of type Σ over the set $\{x_0, \ldots, x_{m-1}\}$. Then $\mathbb{T}(x_0, \ldots, x_{m-1})$ is an absolutely free Σ-algebra freely generated by the set $\{x_0, \ldots, x_{m-1}\}$, see [4] or [6].

We aim to define a mapping $\varrho^{(m)}$ of $\mathbb{T}(x_0, \ldots, x_{m-1})$ into the set P^{P^m} of all maps $P^m \longrightarrow P$, where P is the underlying set of the $\emptyset$-generated absolutely free Σ-algebra $\mathbb{P}$ from 5.1. We write $\pi_i^{(m)}$ to denote the i-th product projection $P^m \longrightarrow P$ and, for each $\sigma \in \Sigma_0$, we let $\sigma^{(m)}$ stand for the constant map $P^m \longrightarrow P$ with the value $p_\sigma \in P_0$.

We define a mapping $\varrho^{(m)}$ recursively as follows.

$$\varrho^{(m)}(x_i) = \pi_i^{(m)} \text{ for } i \in m,$$
$$\varrho^{(m)}(\sigma) = \sigma^{(m)} \text{ for } \sigma \in \Sigma_0, \text{ and}$$
$$\varrho^{(m)}\big(\sigma(t_0,\dots,t_{n-1})\big) = p_\sigma \circ \big(\varrho^{(m)}(t_0)\dot{\times}\cdots\dot{\times}\varrho^{(m)}(t_{n-1})\big) \text{ for } \sigma \in \Sigma_n \text{ with } n \geq 1.$$

It is easily verified that $\varrho^{(m)}\colon \mathbb{T}(x_0,\dots,x_{m-1}) \longrightarrow P^{P^m}$ is one-to-one.

For any given $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, let $\mathbb{T}_\Omega(x_0,\dots,x_{m-1})$ consist of all those members of $\mathbb{T}(x_0,\dots,x_{m-1})$ which have a subterm

$$\sigma'(x_{i_0},\dots,x_{i_{\mathrm{ar}\,\sigma'-1}}) \text{ with } \sigma' \in \Omega \text{ and distinct } x_{i_0},\dots,x_{i_{\mathrm{ar}\,\sigma'-1}} \in \{x_0,\dots,x_{m-1}\}.$$

Denote $H^{(m)} = \varrho^{(m)}\big(\mathbb{T}(x_0,\dots,x_{m-1})\big)$ and $H_\Omega^{(m)} = \varrho^{(m)}\big(\mathbb{T}_\Omega(x_0,\dots,x_{m-1})\big)$.

Now we are ready to state the main result of [7].

M [7]. *If* $\operatorname{card}\Sigma_0 \geq 2^{\aleph_0} + \operatorname{card}(\Sigma\setminus\Sigma_0)$, *then for every* $\Omega \subseteq \Sigma\setminus(\Sigma_0\cup\Sigma_1)$ *there exists a metric* τ_Ω *on the set* P *so that, for the metric space* $X_\Omega = (P,\tau_\Omega)$ *and any integer* $m \geq 1$, *a mapping* $f\colon X_\Omega^m \longrightarrow X_\Omega$ *is continuous exactly when* $f \in H^{(m)} \setminus H_\Omega^{(m)}$. ∎

In this section we prove that the spaces X_Ω constructed in [7] are disciplined, and that they have no absolute fixpoints when $\Sigma\setminus\Sigma_0 \neq \emptyset$. To do this, we need to use some of their structural properties that were stated only implicitly in [7] or had only an auxiliary rôle there.

5.3. As in 3.4 of [7], we write $P_0 = \{p_\sigma \mid \sigma \in \Sigma_0\}$, $B_\sigma = p_\sigma(P^n)$ for $\sigma \in \Sigma_n$ with $n \geq 1$, and $B = \bigcup\{B_\sigma \mid \sigma \in \Sigma\setminus\Sigma_0\} = P \setminus P_0$.

In 3.5 of [7] it is shown that (P,τ_Ω) is a metric space of diameter 1 such that

(α) B is a closed subset of (P,τ_Ω),

(β) p_σ is an isometry $(P^n,\tau_\Omega^n) \longrightarrow (B_\sigma, \tau_\Omega\restriction B_\sigma)$ when $\sigma \in \Sigma_n\setminus\Omega$,

(γ) $\tau_\Omega(B_\sigma,B_{\sigma'}) = 1$ whenever $\sigma,\sigma' \in \Sigma\setminus\Sigma_0$ are distinct,

(δ) p_σ is an isometry $(P^n,\tilde{\tau}_\Omega^n) \longrightarrow (B_\sigma,\tau_\Omega\restriction B_\sigma)$ when $\sigma \in \Sigma_n\cap\Omega$,

where $\tilde{\tau}_\Omega^n$ in (δ) is an auxiliary metric which we describe in 5.4 below.

From (α) and (γ) it follows that every set B_σ with $\sigma \in \Sigma\setminus\Sigma_0$ is closed in X_Ω, and (β) implies that p_σ is a homeomorphism of X_Ω^n onto the closed subset B_σ for any $\sigma \in \Sigma_n\setminus\Omega$.

5.4. For P as in 5.1 and B as in 5.3, let us write

$$P^n[i,j] = \{(z_0,\dots,z_{n-1}) \in P^n \mid z_i = z_j\} \text{ for distinct } i,j \in n,$$
$$P^n[i,c] = \{(z_0,\dots,z_{n-1}) \in P^n \mid z_i = c\} \text{ for } i \in n \text{ and } c \in P,$$
$$P^n[i,B] = \{(z_0,\dots,z_{n-1}) \in P^n \mid z_i \in B\} \text{ for } i \in n.$$

and call these sets *small subsets* of X_Ω^n. It is clear that $P^n[i,j]$ and $P^n[i,c]$ are closed subsets of X_Ω^n, and that $P^n[i,B]$ is a closed subset of X_Ω^n because of 5.3(α).

We need to show that, on any small subset of X_Ω^n, the product metric τ_Ω^n induces the same topology as an auxiliary metric $\tilde{\tau}_\Omega^n$ from [7]. The latter metric is given for $n \geq 2$ in 3.4 of [7] by

$$\tilde{\tau}_\Omega^n(x,y) = \min\{1, \tau_\Omega^n(x,y) + |g(x)-g(y)|\},$$

for a real-valued function $g\colon P^n \longrightarrow [0,1]$ and a point $a^{(n)} = (a_0^{(n)}, \dots, a_{n-1}^{(n)}) \in P_0^n$ satisfying $a_i^{(n)} \neq a_j^{(n)}$ whenever $i, j \in n$ are distinct.

The function g, the point $a^{(n)}$, and certain small subsets of X_Ω^n are related as follows:

(1) g is continuous on $X_\Omega^n \setminus \{a^{(n)}\}$,

(2) $g(x) = 0$ for any $x \in \bigcup\{P^n[i, a_i^{(n)}] \mid i \in n\}$, and also for any x outside a neighbourhood of $a^{(n)}$ which is disjoint with the sets $\bigcup\{P^n[i,j] \mid i \neq j, i, j \in n\}$ and $\bigcup\{P^n[i, B] \mid i \in n\}$,

(3) g is discontinuous at $a^{(n)}$.

Therefore the restriction of g to any small subset of X_Ω^n—including the sets $P^n[i, a_i^{(n)}]$—is continuous, so that $\tilde{\tau}_\Omega^n$ and τ_Ω^n induce the same topology on every small subset of X_Ω^n. This implies that the identity map of $X_\Omega^n = (P^n, \tau_\Omega^n)$ onto $(P^n, \tilde{\tau}_\Omega^n)$ sends each small subset homeomorphically onto a closed subset of $(P^n, \tilde{\tau}_\Omega^n)$. Since each B_σ with $\sigma \in \Sigma \setminus \Sigma_0$ is closed, 5.3(δ) implies that, for any $\sigma \in \Sigma_n \cap \Omega$, the restriction of p_σ to any small subset of X_Ω^n is a homeomorphism onto a closed subset of X_Ω.

5.5. Here is a summary of earlier observations which will be needed in Proposition 5.6 and in Section 7:

(1) if $\sigma \in \Sigma_n \setminus \Omega$ and $n \geq 1$, then p_σ is a homeomorphism of X_Ω^n onto a closed subset of X_Ω,

(2) if $\sigma \in \Sigma_n \cap \Omega$, then p_σ is not continuous, but its restriction to any small subset of X_Ω^n is a homeomorphism onto a closed subset of X_Ω,

(3) the topology of $(P^n, \tilde{\tau}_\Omega^n)$ is finer than the topology of (P^n, τ_Ω^n) for every $n \geq 2$.

5.6. PROPOSITION. *The space X_Ω is disciplined for every $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$.*

PROOF. Write $X = X_\Omega$. Since X is metrizable, we need only show that, for any $m \geq 1$, every non-constant continuous map $f\colon X^m \longrightarrow X$ is of the form $f = g \circ \pi^{(M)}$, where $\pi^{(M)}\colon X^m \longrightarrow X^M$ is the projection associated with some $M \subseteq m$, and g is a homeomorphism of X^M onto a closed subset of X.

Let $m \geq 1$ and let $f\colon X^m \longrightarrow X$ be non-constant and continuous. According to 5.2, there is a unique term $t_f \in \mathbb{T}(x_0, \dots, x_{m-1}) \setminus \mathbb{T}_\Omega(x_0, \dots, x_{m-1})$ with $\varrho^{(m)}(t_f) = f$.

We use the term recursivity of the absolutely free Σ-algebra $\mathbb{T}(x_0, \dots, x_{m-1})$ as follows.

If $t_f = x_i$ with $i \in m$, then $f = \pi_i^{(m)}$ is the i-th product projection $X^m \longrightarrow X$, and hence it has a required decomposition, namely $f = \mathrm{id}_X \circ \pi_i^{(m)}$. Otherwise $t_f = \sigma(t_0, \dots, t_{n-1})$ for some $\sigma \in \Sigma$ and $t_0, \dots, t_{n-1} \in \mathbb{T}(x_0, \dots, x_{m-1}) \setminus \mathbb{T}_\Omega(x_0, \dots, x_{m-1})$. Since f is non-constant, we must have $\sigma \in \Sigma_n$ with $n \geq 1$, and $f = p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{n-1})$, where $f_j = \varrho^{(m)}(t_j)$ for all $j \in n$ and at least one of the maps $f_j\colon X^m \longrightarrow X$ is non-constant. Assuming that

> for every $j \in n$, there is some $M_j \subseteq m$ and some homeomorphism g_j of X^{M_j} onto a closed subset of X such that $f_j = g_j \circ \pi^{(M_j)}$,

(where we write $M_j = \emptyset$, $\pi^{(M_j)}\colon X^m \longrightarrow X^\emptyset$ and $g_j\colon X^\emptyset \longrightarrow X$ when f_j is a constant map), we shall construct a required decomposition $f = g \circ \pi^{(M)}$.

First of all, we note that $\bar{g} = \prod\{g_j \mid j \in n\}$ is a homeomorphism of $\prod\{X^{M_j} \mid j \in n\}$ onto a closed subset of X^n. Set $M = \bigcup\{M_j \mid j \in n\}$, and let $\psi_j\colon M_j \longrightarrow M$ denote the inclusion map. For any $\varphi \in X^M$ write $h(\varphi) = (\varphi \circ \psi_0, \dots, \varphi \circ \psi_{n-1})$. Then

$$h: X^M \longrightarrow \prod\{X^{M_j} \mid j \in n\}$$

is a homeomorphism of X^M onto a closed subset of $\prod\{X^{M_j} \mid j \in n\}$.

If $\Delta: X^m \longrightarrow (X^m)^n$ is the 'diagonal map' defined by $\Delta(\varphi) = (\varphi, \dots, \varphi)$ for all $\varphi \in X^m$, and if $\pi = \prod\{\pi^{(M_j)} \mid j \in n\}$, then $\pi \circ \Delta = h \circ \pi^{(M)}$. Since $\tilde{f} = f_0 \dot{\times} \cdots \times f_{n-1}$ has a decomposition

$$X^m \xrightarrow{\Delta} (X^m)^n \xrightarrow{\pi} \prod\{X^{M_j} \mid j \in n\} \xrightarrow{\bar{g}} X^n,$$

it follows that $\tilde{f} = (\bar{g} \circ h) \circ \pi^{(M)}$, where $\bar{g} \circ h$ is a homeomorphism of X^M onto a closed subset of X^n.

For $n \geq 1$ and $\sigma \in \Sigma_n \setminus \Omega$, the mapping p_σ is a homeomorphism of X^n onto a closed subset of X, by 5.5(1), so that the composite $g = p_\sigma \circ \bar{g} \circ h$ is a homeomorphism of X^M onto a closed subset of X again. Therefore $f = p_\sigma \circ \tilde{f} = g \circ \pi^{(M)}$ has a required decomposition.

Let $\sigma \in \Sigma \cap \Omega$. Then $\sigma \in \Sigma_n \cap \Omega$ for some $n \geq 2$, and we show that $\tilde{f}(X^m)$ is contained in a small subset of X^n. Indeed, if at least one of the component maps f_j of $\tilde{f} = f_0 \dot{\times} \cdots \dot{\times} f_{n-1}$ is the constant with the value c, then $\tilde{f}(X^m) \subseteq P^n[j, c]$. Assume that no map f_j with $j \in n$ is constant. If one of them, say f_{j_0}, is not a product projection, then $f_{j_0} = p_{\sigma'} \circ h$ for some $h: X^m \longrightarrow X^{\operatorname{ar} \sigma'}$ and $\sigma' \in \Sigma \setminus \Sigma_0$, and hence $f(P^m) \subseteq P^n[j_0, B]$. In the remaining case, every f_j is a projection $\pi_{i_j}^{(m)}: X^m \longrightarrow X$, so that $\tilde{f}(x_0, \cdots, x_{m-1}) = (x_{i_0}, \dots, x_{i_{n-1}})$. Should $x_{i_0}, \dots, x_{i_{n-1}}$ be pairwise distinct, then $\sigma(x_{i_0}, \dots, x_{i_{n-1}}) \in \mathbb{T}_\Omega(x_0, \dots, x_{m-1})$ and hence f would not be continuous. Therefore $i_r = i_s$ for some distinct $r, s \in n$, and hence $f(P^m) \subseteq P^n[i_r, i_s]$. Altogether, $\tilde{f}(P^m)$ is always contained in a small subset of X^n. Since $\tilde{f}(P^m)$ is closed in X^n, it is also closed in any small subset containing it. By 5.5(2), the restriction of p_σ to $\tilde{f}(P^m)$ maps the set $\tilde{f}(P^m) = \bar{g}h(X^M)$ homeomorphically onto a closed subset of X, so that $g = p_\sigma \circ \bar{g} \circ h$ is a homeomorphism of X^M onto a closed subset of X and $f = p_\sigma \circ \tilde{f} = g \circ \pi^{(M)}$ also in this case.

Therefore every continuous non-constant $f: X^m \longrightarrow X$ with $m \geq 1$ has a required decomposition. ∎

5.7. LEMMA. *If $\Sigma \setminus \Sigma_0 \neq \emptyset$ and $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ is arbitrary, then there exists a continuous map $s: X_\Omega \longrightarrow X_\Omega$ such that $s \circ f \neq f$ for every $f: X_\Omega^k \longrightarrow X_\Omega$ with $k \geq 0$. In particular, the space X_Ω has no absolute fixpoints.*

PROOF. If $\sigma \in \Sigma_n$ and $n \geq 1$, then either $n = 1$ and $\sigma \notin \Omega$, or else $n \geq 2$ and the diagonal of X_Ω^n is a closed subset of the small subset $P^n[0, 1] \subset X_\Omega^n$. Hence $s(x) = p_\sigma(x, \dots, x)$ is a homeomorphism of X_Ω onto a closed subset of B_σ for any $\sigma \in \Sigma \setminus \Sigma_0$. If $s(a) = a$ for some $a \in P$ then $\sigma(a, \dots, a) = a$ in the absolutely free $\emptyset$-generated Σ-algebra $\mathbb{P}$, which is impossible. Therefore $s \circ f \neq f$ for every continuous $f: X_\Omega^k \longrightarrow X_\Omega$ with $k \geq 0$. ∎

5.8. COROLLARY. *If $\Sigma \setminus \Sigma_0 \neq \emptyset$ and $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, then X_Ω is a disciplined space without absolute fixpoints.* ∎

6. **Operation clones of traces.**

6.1. To complete the proof of Theorem 1.9, we recall that the trace $\mathbb{A}(X) = (Q, Z \cup \{\cdot\})$ of any infinite metrizable space $X = (P, t)$ has a binary operation $\cdot$ and a countable set $Z = \{\gamma\} \cup \{\pi_j \mid j \in \omega\}$ of unary operations on the set $Q = S \cup P \cup \{\lambda\}$, in which S is the collection of all one-to-one sequences in P. One can easily verify that the binary operation $\cdot$ and the set Z of the unary operations satisfy these identities:

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot y = y \cdot x$
(2) $\sigma(x \cdot y) = x \cdot y$ for all $\sigma \in Z$,
(3) $\rho\sigma(x) = \sigma(x)$ for all $\rho, \sigma \in Z$,
(4) $\sigma(x) \cdot \sigma(x) = \sigma(x)$ for all $\sigma \in Z$,
(5) $\pi_0(x) \cdot \pi_1(x) \cdot y = x \cdot y$,
(6) $\pi_j(x) \cdot \pi_{j'}(x) \cdot t(x) = \pi_0(x) \cdot \pi_1(x)$ for all $j \neq j'$ in ω, provided $t(x) = x$, or $t(x) = \sigma(x)$ for some $\sigma \in Z$, or $t(x)$ is missing altogether.

6.2. PROPOSITION. *Let $X = (P, t)$ be an infinite metrizable space with no isolated points. Then an identity holds in $\mathbb{A}(X)$ if and only if it is a consequence of the identities described in (1)–(6) above.*

PROOF. For $n, n' \geq 1$, let $t(x_1, \ldots, x_n)$ and $t'(x'_1, \ldots, x'_{n'})$ be terms in the basic operations of $\mathbb{A}(X)$ over the variables indicated. We need to show that any identity

$$\text{(I)} \qquad t(x_1, \ldots, x_n) = t'(x'_1, \ldots, x'_{n'})$$

either fails to hold in $\mathbb{A}(X)$ or else follows from (1)–(6).

The following easily established properties of $\mathbb{A}(X)$ will be used:

(a) if $t(x_1, \ldots, x_m)$ is a term and $p \in P$, then $t(p, \ldots, p) = p$,
(b) if $q, q' \in Q$ and $q \cdot q' \neq \lambda$, then $q = q' = q \cdot q' \in P$,
(c) if $t(x_1, \ldots, x_n)$ is a term and $(q_1, \ldots, q_n) \in Q^n$ with $q_i = \lambda$ for some $i = 1, \ldots, n$, then $t(q_1, \ldots, q_n) = \lambda$.

We claim that an identity (I) is satisfied in $\mathbb{A}(X)$ only when it is regular, that is, when it has the form

$$\text{(R)} \qquad t(x_1, \ldots, x_n) = t'(x_1, \ldots, x_n).$$

If not, then, say, $x_1 \notin \{x'_1, \ldots, x'_{n'}\}$. Select any $p \neq p'$ in P and substitute p for x_1 and p' for all other variables occurring in (I). Then $t'(p', \ldots, p') = p'$ and $t(p, p', \ldots p') \in \{p, \lambda\}$, by (a), (b) and (c). Since $p' \notin \{p, \lambda\}$, such an identity fails to hold in any $\mathbb{A}(X)$.

Before discussing reasons why (R) holds or fails in $\mathbb{A}(X)$, we note that any term t in which $\cdot$ does not occur is either a variable or $t(x) = \sigma(x)$ holds in $\mathbb{A}(X)$ for some $\sigma \in Z$, by (3).

CASE 1. Suppose that $\cdot$ occurs on neither side of (R). Since members of Z act differently from one another and from the identity map on the trace $\mathbb{A}(X)$ of any infinite X, the identity (R) holds in $\mathbb{A}(X)$ exactly when it follows from (1)–(6).

We turn to the remaining possibilities now.

Suppose that $\cdot$ does occur in $t(x_1, \dots, x_n)$. Then (1)–(3) imply that $t = t_1 \cdot \ldots \cdot t_r$ where each t_i is either one of the variables of t, or $t_i(x) = \sigma(x)$ holds in $\mathbb{A}(X)$ for some $\sigma \in Z$. Having applied (5) twice to each t_i which is a variable x, we find that $t_i(x) = \pi_0(x) \cdot \pi_1(x)$ holds in $\mathbb{A}(X)$, and may henceforth assume that every t_i has the form $t_i(x) = \sigma(x)$ for some $\sigma \in Z$. Then, using (4) to eliminate all repetitions, we may also assume that $t_i \neq t_j$ whenever $i \neq j$. Having applied (6), we also conclude that each variable x needs to occur in t at most twice, and that $t(x) = t_x(x)$ if t is unary or else $t(x) = t_x(x) \cdot T_x$—where the variables of T_x are the variables of t other than x, and $t_x(x)$ is one of the unary terms

$$\text{(U)} \qquad \pi_0(x) \cdot \pi_1(x), \gamma(x), \text{ or } \pi_j(x) \cdot \gamma(x), \pi_j(x) \text{ with } j \in \omega$$

in either case.

In particular, for any unary term t in which $\cdot$ does occur, in any $\mathbb{A}(X)$ we have either $t(x) = \pi_0(x) \cdot \pi_1(x)$ or $t(x) = \pi_j(x) \cdot \gamma(x)$ for some $j \in \omega$.

For a future use we note that, since an infinite disciplined metrizable space has no isolated points, for any $p \in P$ there exists a one-to-one sequence s converging to p. In particular, there are sequences $s, s_j \in S$ converging to p such that $\pi_j(s) \neq p$ for all $j \in \omega$, and $\pi_k(s_j) = p$ exactly when $k = j \in \omega$. In algebraic terms, this respectively means that $\pi_j(s) \cdot \gamma(s) = \lambda$ for all $j \in \omega$, and that $\pi_k(s_j) \cdot \gamma(s) \neq \lambda$ if and only if $k = j \in \omega$.

Case 2. Suppose that $\cdot$ occurs in t but not in t'. Thus $t'(x) = x$ or $t'(x) = \sigma(x)$ for some $\sigma \in Z$. We claim that (R) fails to hold in any $\mathbb{A}(X)$. In the first case, $t(s) \in P \cup \{\lambda\} \not\ni s = t'(s)$ for any $s \in S$. In the second, we choose a convergent sequence $s \in S$ so that $\pi_j(s) \neq \gamma(s)$ for all $j \in \omega$. Then $t(s) = \lambda \notin P$ and $t'(s) = \sigma(s) \in P$ for every $\sigma \in Z$. Whence (R) fails to hold in $\mathbb{A}(X)$ in either case.

Case 3. Suppose that $\cdot$ occurs in both t and t'. Let the identity (R) hold in $\mathbb{A}(X)$, and let x be one of its variables. Then $t = t_x(x) \cdot T_x$ and $t' = t'_x(x) \cdot T'_x$—where x occurs neither in T_x nor in T'_x and $t_x(x)$, $t'_x(x)$ are amongst the unary terms listed in (U).

(3a) Suppose that $t_x(x) = \sigma(x)$ for some $\sigma \in Z$. To show that $t_x(x) = t'_x(x)$, we select an $s \in S$ so that $\gamma(s) \in P$ and $\gamma(s) \neq \pi_j(s)$ for all $j \in \omega$, and then substitute s for x and $\sigma(s)$ for each of the remaining variables in (R). Then $\sigma(s) \in P$ and, if T_x and T'_x are present, also $T_x = T'_x = \sigma(s)$, by (a). If (R) is a unary identity, then $\sigma(s) = t'_x(s)$. If not, then $\sigma(s) = t = t'_x(s) \cdot \sigma(s)$ by (4), and hence $\sigma(s) = t'_x(s)$ follows again, this time from (b). Since s is one-to-one and $\gamma(s) \neq \pi_j(s)$, we have $\pi_0(s) \cdot \pi_1(s) = \lambda = \pi_j(s) \cdot \gamma(s)$ for all $j \in \omega$, and also $\sigma'(s) \neq \sigma(s)$ in P whenever $\sigma, \sigma' \in Z$ are distinct. Since $t'_x(x)$ is one of the terms listed in (U), the only remaining possibility is that $t'_x(x) = \sigma(x)$. But then $t_x(x) = t'_x(x)$, as claimed.

(3b) To show that $t_x(x) = t'_x(x)$ in all cases, suppose now that $t_x(x) = \pi_j(x) \cdot \gamma(x)$ for some $j \in \omega$. Select $s_j \in S$ so that $\gamma(s_j) = \pi_j(s_j)$, and substitute s_j for x and $\pi_j(s_j) \in P$ for all other variables in (R). Then $t_x(s_j) = \pi_j(s_j) \in P$, and $T_x = T'_x = \pi_j(s_j)$—if these terms are present. Then $\pi_j(s_j) = t'_x(s_j)$ follows as in (3a), from where we also already know that $t'_x(x)$ cannot be $\sigma(x)$ for any $\sigma \in Z$. If $t'_x(x) = \pi_0(x) \cdot \pi_1(x)$, then $t'_x(s_j) = \lambda \notin P$ because s_j is one-to-one, and (R) fails. The remaining possibility given by (U) is that

$t'_x(x) = \pi_k(x) \cdot \gamma(x)$ for some $k \in \omega$. But $t'_x(s_j) = \pi_k(s_j) \cdot \gamma(s_j) = \pi_k(s_j) \cdot \pi_j(s_j) \in P$ only when $k = j$ because of (b) and the fact that s_j is one-to-one. Therefore $t_x(x) = t'_x(x)$ whenever neither term is equal to $\pi_0(x) \cdot \pi_1(x)$ in $\mathbb{A}(X)$. This concludes the proof that $t_x(x) = t'_x(x)$ follows for every variable x of any identity (R) which holds in $\mathbb{A}(X)$.

Altogether, any identity (I) either follows from (1)–(6) or else fails to hold in every $\mathbb{A}(X)$. ∎

The proof of Theorem 1.9 is now complete.

7. Consequences and modifications.

7.1 . Let $(\mathcal{K}, \mathrm{U})$ be a concrete category with finite products preserved by its forgetful functor U. Then the clone $\mathrm{Clo}(a)$ exists for every object a of $\mathcal{K}$ with non-empty underlying set. If b is another such object, then the suprema

$$i(a,b) = \sup\{k+1 \mid \mathrm{Clo}_k(a) \simeq \mathrm{Clo}_k(b)\} \text{ and}$$
$$e(a,b) = \sup\{k+1 \mid \mathrm{Clo}_k(a) \approx \mathrm{Clo}_k(b)\}$$

exist and their values belong to the set $\{1, 2, \ldots, \infty\}$. It is clear that the two resulting functions satisfy

(1) $\varphi(a,a) = \infty$,
(2) $\varphi(a,b) = \varphi(b,a)$, and
(3) $\varphi(b,c) \geq \min\{\varphi(a,b), \varphi(a,c)\}$.

This leads to a natural question about the representability of φ by i or e over some category $\mathcal{K}$ of universal algebras. The question is addressed in Theorems 7.2–3 below. These follow from Theorems 1 and 2 of [7]—now valid for infinite disciplined metrizable spaces without absolute fixpoints—by means of Propositions 3.5 and 4.1.

REMARK. For the reciprocal $\rho = 1/\varphi$ of a function φ satisfying (1)–(3) we have

$$\rho(a,a) = 0,\ \rho(a,b) = \rho(b,a) \text{ and } \rho(b,c) \leq \max\{\rho(a,b), \rho(a,c)\},$$

which are the axioms of a non-archimedean pseudometric. Every metrizable 0-dimensional topological space can be metrized by a non-archimedean metric ρ with values in the set of integer reciprocals, see [2]. Theorem 7.2 below implies that any such pseudometric space can be represented by the reciprocal $1/i$ of our function i on a suitable collection of universal algebras.

7.2. THEOREM. *Let C be a set, and let $\varphi\colon C \times C \to \{1, 2, \ldots, \infty\}$ be a function satisfying 7.1(1)–(3). Then there exists a system $\{A_c \mid c \in C\}$ of finitary algebras with pairwise isomorphic clones of operations, and such that $i(A_a, A_b) = \varphi(a,b)$ for every $(a,b) \in C \times C$. Moreover, if $\varphi(a,b) > 1$ for all $(a,b) \in C$, then all algebras A_c can be constructed over the same underlying set.* ∎

Since there are only countably many formulas in the first order language of clones and their k-segments, a result by Erdős and Radó [3] implies that an analogue of Theorem 7.2 for elementary equivalence cannot hold when card $C > 2^{\aleph_0}$. In Theorem 7.3 below we present a partial positive result that leaves open the remaining case of $\aleph_0 <$ card $C \leq 2^{\aleph_0}$.

7.3. THEOREM. *Let C be a countable set, and let $\varphi: C \times C \longrightarrow \{1, 2, \dots, \infty\}$ be a function satisfying 7.1(1)–(3). Then there is a system $\{B_c \mid c \in C\}$ of finitary algebras with pairwise isomorphic clones of operations, and such that $e(B_a, B_b) = \varphi(a, b)$ for every $(a, b) \in C \times C$. Moreover, if $\varphi(a, b) > 1$ for all $(a, b) \in C$, then all algebras B_c can be constructed over the same underlying set.* ∎

7.4. CONCLUDING REMARKS. Other conclusions can be easily obtained by various manipulations of the algebraic trace $\mathbb{A}(X)$.

(A) Theorems 7.2, 7.3 and 1.9—without their claims about operation clone isomorphism—hold also for algebras of other similarity types.

(A1) Countably many unary operations suffice. To see this, for any infinite disciplined metrizable space $X = (P, t)$ with no absolute fixpoints, we simply replace the trace $\mathbb{A}(X) = (Q, Z \cup \{\cdot\})$ by an algebra $\mathbb{B}(X) = (Q \times Q, \{\sigma_2 \mid \sigma \in Z\} \cup \{\beta, P_0, P_1\})$ in which, for all $(x_0, x_1) \in Q \times Q$, we set $\sigma_2(x_0, x_1) = \big(\sigma(x_0), \sigma(x_1)\big)$ when $\sigma \in Z$, $\beta(x_0, x_1) = (x_0 \cdot x_1, x_0 \cdot x_1)$ and $P_i(x_0, x_1) = (x_i, x_i)$ for $i \in 2$. Then $\mathbb{B}(X)$ is a unary algebra such that $g: \mathbb{B}(X)^n \longrightarrow \mathbb{B}(X)$ is a homomorphism exactly when $g = f \times f$ for some homomorphism $f: \mathbb{A}(X)^n \longrightarrow \mathbb{A}(X)$.

(A2) A single ω-ary operation suffices as well. For any infinite disciplined metrizable space $X = (P, t)$ with no absolute fixpoints, we replace $\mathbb{A}(X)$ by an algebra $\mathbb{C}(X) = (R, \Gamma)$ of type $\{\omega\}$ in which $R = P \cup \{\lambda\}$ and, for any $s \in R^\omega$,

$$\Gamma(s) = \begin{cases} p & \text{if } s \in P^\omega \text{ converges to } p \\ \lambda & \text{in all other cases.} \end{cases}$$

It is clear that, for $n \geq 0$, any homeomorphism h of X^n onto a closed subset of X extends to a homomorphism $E(h): \mathbb{C}(X)^n \longrightarrow \mathbb{C}(X)$, and that $E(h)$ is constant whenever h is. Proposition 2.3 is, in fact, somewhat easier to prove for $\mathbb{C}(X)$ than for $\mathbb{A}(X)$, and the remainder is straightforward.

(A3) In a forthcoming paper, it will be shown that three unary operations are also sufficient. The proof uses a modification of a type-reducing construction from [5].

(B) Theorem 1.9 and its unary forms hold also for an algebra and its reduct—in place of two algebras with isomorphic operation clones.

The spaces X_1 and X_2 used to prove Theorem 1 in [7], of which Theorem 1.9 is an algebraic translation by means of their traces $\mathbb{A}(X_i)$, have the same underlying set and satisfy $\mathrm{Clo}(X_1) \subset \mathrm{Clo}(X_2)$, see also 5.5(3). The algebras $\mathbb{A}(X_1)$ and $\mathbb{A}(X_2)$ inherit these properties. Hence $\mathbb{A}(X_1)$ can be expanded to an algebra $\mathbb{D}(X_1)$ by adding a new unary operation δ_1 defined by $\delta_1(x) = \gamma_2(x)$, where γ_2 was the unary 'convergence' operation of $\mathbb{A}(X_2)$. Then $\mathrm{Clo}\big(\mathbb{D}(X_1)\big) = \mathrm{Clo}\big(\mathbb{A}(X_1)\big)$ follows from Proposition 2.3 and the fact that $\mathrm{Clo}(X_1) \subset \mathrm{Clo}(X_2)$. The reduct $\mathbb{D}(X_2) = \mathbb{A}(X_2)$ is then obtained from $\mathbb{D}(X_1)$ by removing the original 'convergence' operation γ_1 of $\mathbb{A}(X_1)$.

REFERENCES

1. G. Birkhoff and J. D. Lipson, *Heterogeneous algebras*, J. Comb. Theory **8**(1970), 115–133.
2. J. de Groot, *Non-archimedean metrics in topology*, Proc. Amer. Math. Soc. **17**(1956), 948–953.
3. P. Erdős and R. Radó, *A combinatorial theorem*, J. London Math. Soc. **25**(1950), 249–255.
4. G. A. Grätzer, *Universal Algebra*, 2nd Edition, Springer-Verlag, 1979.
5. R. McKenzie, *A new product of algebras and a type reduction theorem*, Algebra Universalis **8**(1984), 26–69.
6. R. McKenzie, G. McNulty and W. Taylor, *Algebras, lattices, varieties*, Volume 1, Brooks-Cole, Monterey, California, 1987.
7. J. Sichler and V. Trnková, *Isomorphism and elementary equivalence of clone segments*, Period. Math. Hungar. **32**(1996), 113–128.
8. W. Taylor, *The clone of a topological space*, Research and Exposition in Mathematics, vol. 13, Heldermann Verlag, 1986.
9. V. Trnková, *Semirigid spaces*, Trans. Amer. Math. Soc. **343**(1994), 305–325.
10. ———, *Algebraic theories, clones and centralizers*, preprint, 1994.

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THE REPRESENTATION RING OF THE TWISTED QUANTUM DOUBLE OF A FINITE GROUP

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ABSTRACT. We provide an isomorphism between the Grothendieck ring of modules of the twisted quantum double of a finite group, and a product of centres of twisted group algebras of centralizer subgroups. It follows that this Grothendieck ring is semisimple. Another consequence is a formula for the characters of this ring in terms of representations of twisted group algebras of centralizer subgroups.

Introduction. Let G be a finite group and k an algebraically closed field. Given a 3-cocycle $\omega\colon G \times G \times G \longrightarrow k^{\times}$, there is a quasi-Hopf algebra $D^{\omega}(G)$ with underlying vector space $(kG)^{*} \otimes_{k} kG$, where $(kG)^{*}$ is the Hopf algebra dual to the group algebra kG [1, 4, 7]. As an algebra, $D^{\omega}(G)$ is a crossed product of the Hopf algebras $(kG)^{*}$ and kG. There is also a construction of this algebra from the ordinary quantum double $D(G)$, originally defined by Drinfel'd [9], analogous to that of a twisted group algebra from a group algebra. We have therefore chosen to call $D^{\omega}(G)$ the twisted quantum double of G.

In the case $k = \mathbf{C}$, it is indicated by Dijkgraaf, Pasquier, and Roche in [7] that there is a connection between the representations of $D^{\omega}(G)$ and conformal field theory; in this context, tensor products of these representations are of interest. Applications of such representations to generalized Thompson series and Moonshine are given in [1] by Bantay. In [14], Mason discusses a conjectured equivalence of categories between $D^{\omega}(G)$-modules and modules of a certain vertex operator algebra, and in [8], Dong and Mason prove a special case of this conjecture.

The algebra $D^{\omega}(G)$ turns out to be an example of a quasitriangular quasi-Hopf algebra. For our purposes, this implies that there is a representation ring $R\big(D^{\omega}(G)\big)$ which is both associative and commutative. The sum and product of this ring are given by direct sum and tensor product of representations. In this exposition, we will analyze the structure of this ring, or more precisely of the Grothendieck ring $\mathcal{R}\big(D^{\omega}(G)\big)$, the quotient of $R\big(D^{\omega}(G)\big)$ by the ideal of short exact sequences.

In Section 1, we give an explicit definition of $D^{\omega}(G)$, and discuss some of its properties. In particular, we show that $D^{\omega}(G)$ is semisimple if and only if the characteristic p of

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k does not divide the order of G. In the non-semisimple case, the Grothendieck ring is a proper quotient of the representation ring, and is easier to study. In Section 1 we also state a characterization of indecomposable $D^\omega(G)$-modules, which is a simultaneous generalization of the case $k = \mathbf{C}$ in [7], and of the ordinary quantum double $D(G)$ over an arbitrary field in [14].

Our main result (Theorem 4.4) describes the structure of the Grothendieck ring $\mathcal{R}(D^\omega(G))$, and implies that it is semisimple. More precisely, we show that there is an isomorphism from $\mathcal{R}(D^\omega(G))$ to the product of centres of twisted group algebras of centralizer subgroups of G. As a consequence, we give its set of characters, that is algebra homomorphisms from $\mathcal{R}(D^\omega(G))$ to $\mathbf{C}$. These characters distinguish modules up to their composition factors, and are analogous to Brauer characters of the group, which correspond to the characters of the Grothendieck ring $\mathcal{R}(kG)$ of the group algebra kG [2]. They are expressed in terms of characters of twisted group algebras of centralizer subgroups of G, that is trace functions of modules for these algebras. Such modules correspond to projective representations of the centralizer subgroups. Our theorem generalizes a result for the ordinary quantum double $D(G)$ given in [16].

We first present our results in the special case $k = \mathbf{C}$ in Section 2. In this case the Grothendieck ring described above is equal to the representation ring, and proofs are straightforward. For the case of an arbitrary algebraically closed field k, we start with the same approach. However, now $D^\omega(G)$ is an algebra over k, whereas its representation ring is an algebra over $\mathbf{C}$. The situation is complicated by the necessity of lifting certain function values from k to $\mathbf{C}$. It also appears necessary to introduce a representation group, that is a covering group such that any projective representation (or representation of a twisted group algebra) may be lifted to an ordinary representation of this group. In Section 3, we present an analog of Brauer characters for twisted group algebras, and results relating these Brauer characters to those of representation groups. In Section 4, we use these results to prove our main theorem.

All our modules will be finite dimensional over k, and all tensor products will be over k unless otherwise indicated. We caution that we will use the term "character" in two different ways. At times, it will refer to a homomorphism from an algebra to $\mathbf{C}$. At other times, it will refer to the trace function of a module. In [3], Benson and Parker deal with this confusion by referring to the former as a "species." Here we hope that it will be clear from context which meaning is intended.

1. **Preliminaries.** We will define the twisted quantum double as in [1, 4, 7], although we do not require the underlying field to be $\mathbf{C}$. Fix a finite group G and an algebraically closed field k of characteristic p. Let $\omega: G \times G \times G \longrightarrow k^\times$ be a 3-cocycle; that is

$$\omega(a,b,c)\omega(a,bc,d)\omega(b,c,d) = \omega(ab,c,d)\omega(a,b,cd)$$

for all $a, b, c, d \in G$. We assume that ω is normalized so that $\omega(a,b,c)$ is equal to 1 whenever one of a, b, or c is equal to the identity element 1 of G. The *twisted quantum double* $D^\omega(G)$ of G with respect to ω is the following quasi-Hopf algebra with underlying vector space $(kG)^* \otimes_k kG$. (For the definition of a quasi-Hopf algebra, see [4] or [10].) Let $\{\delta_g \otimes \bar{x}\}_{g,x\in G}$ be the canonical basis, where δ_g is the function dual to $g \in G$, so that $\delta_g(g) = 1$ and $\delta_g(h) = 0$ when $h \in G$, $h \neq g$. We will abbreviate $\delta_g \otimes \bar{x}$ by $\delta_g\bar{x}$.

The product is given by

$$(\delta_g\bar{x})(\delta_h\bar{y}) = \theta_g(x,y)\delta_g\delta_{xhx^{-1}}\overline{xy} = \begin{cases} \theta_g(x,y)\delta_g\overline{xy} & \text{if } g = xhx^{-1} \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}gxy)}{\omega(x,x^{-1}gx,y)}$. The element $1_{D^\omega(G)} = \sum_{g\in G}\delta_g\bar{1}$ is the multiplicative identity. We point out that this is the algebra structure given by the crossed product [15] of the Hopf algebras $(kG)^*$ and kG with respect to the conjugation action of G on $(kG)^*$ and the linear map $\sigma\colon kG \otimes kG \longrightarrow (kG)^*$ defined by

$$\sigma(x \otimes y) = \sum_{g\in G} \theta_g(x,y)\delta_g$$

for all $x, y \in G$.

When we use the notation δ_g, we will mean $\delta_g\bar{1}$, and by $\bar{x}$ we will mean $\sum_{g\in G}\delta_g\bar{x}$. Thus the elements $\{\delta_g\}_{g\in G}$ generate a subalgebra of $D^\omega(G)$ isomorphic to $(kG)^*$, but the elements $\{\bar{x}\}_{x\in G}$ do not in general generate a subalgebra isomorphic to kG. However, the elements $\bar{x}$ are invertible, with $\bar{x}^{-1} = \sum_{g\in G}\theta_g(x^{-1},x)^{-1}\delta_g\overline{x^{-1}}$.

The coproduct $\Delta\colon D^\omega(G) \longrightarrow D^\omega(G) \otimes D^\omega(G)$ is given by

$$\Delta(\delta_g\bar{x}) = \sum_{h\in G} \gamma_x(h, h^{-1}g)(\delta_h\bar{x}) \otimes (\delta_{h^{-1}g}\bar{x}),$$

where $\gamma_x(h,\ell) = \frac{\omega(h,\ell,x)\omega(x,x^{-1}hx,x^{-1}\ell x)}{\omega(h,x,x^{-1}\ell x)}$. This coproduct is not coassociative in general, but quasi-coassociative; this means that there is an invertible element

$$\phi = \sum_{g,h,k\in G} \omega(g,h,k)\delta_g\bar{1} \otimes \delta_h\bar{1} \otimes \delta_k\bar{1}$$

in $D^\omega(G)^{\otimes 3}$ such that $(\Delta \otimes \mathrm{id})\Delta(a) = \phi(\mathrm{id} \otimes \Delta)\Delta(a)\phi^{-1}$ for all $a \in D^\omega(G)$.

A counit ϵ and a coinverse s are given by

$$\epsilon(\delta_g\bar{x}) = \delta_{g,1} \quad \text{and} \quad s(\delta_g\bar{x}) = \theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}\delta_{x^{-1}g^{-1}x}\overline{x^{-1}},$$

where $\delta_{g,1}$ is the Kronecker delta. These maps make the algebra $D^\omega(G)$ into a quasi-Hopf algebra [1, 4, 7]. Verification of the details involves the following identities, which result from the 3-cocycle identity for ω:

$$\begin{gathered} \theta_z(a,b)\theta_z(ab,c) = \theta_{a^{-1}za}(b,c)\theta_z(a,bc), \\ \theta_y(a,b)\theta_z(a,b)\gamma_a(y,z)\gamma_b(y^a,z^a) = \theta_{yz}(a,b)\gamma_{ab}(y,z), \\ \gamma_z(a,b)\gamma_z(ab,c)\omega(a^z,b^z,c^z) = \gamma_z(b,c)\gamma_z(a,bc)\omega(a,b,c), \end{gathered}$$

for all $a, b, c, y, z \in G$. If ω is trivial, we obtain the ordinary quantum double $D(G)$ of G, which is a Hopf algebra. The quantum double construction was originally defined by Drinfel'd for any Hopf algebra [9].

We will next define the representation ring. Let U and V be right $D^\omega(G)$-modules. Then $U\otimes V$ is a right $D^\omega(G)$-module via the pullback of the natural action of $D^\omega(G)\otimes D^\omega(G)$ on $U\otimes V$ with respect to the coproduct $\Delta\colon D^\omega(G)\longrightarrow D^\omega(G)\otimes D^\omega(G)$. This results in a right $D^\omega(G)$-module since Δ is an algebra homomorphism. The field k is a right $D^\omega(G)$-module via the pullback of the action of k on itself by right multiplication with respect to the counit $\epsilon\colon D^\omega(G)\longrightarrow k$. Up to isomorphism, this *trivial* module k is a multiplicative identity with respect to tensor product of modules; this follows from the counit property of a quasi-Hopf algebra.

Let $R\big(D^\omega(G)\big)$ denote the *representation ring* of $D^\omega(G)$, that is the $\mathbf{C}$-algebra generated by isomorphism classes of finite dimensional right $D^\omega(G)$-modules with direct sum for addition and tensor product for multiplication. Then $R\big(D^\omega(G)\big)$ is a ring with identity given by the isomorphism class of the trivial module. Associativity of $R\big(D^\omega(G)\big)$ follows from the quasi-coassociativity of the coproduct for $D^\omega(G)$. By abuse of language and notation, we will consider $D^\omega(G)$-modules to be elements of the representation ring, when we really mean their isomorphism classes.

We now define $R_0\big(D^\omega(G)\big)$, the *ideal of short exact sequences*, to be the ideal generated by all $U-U'-U''$ where $0\longrightarrow U'\longrightarrow U\longrightarrow U''\longrightarrow 0$ is a short exact sequence of $D^\omega(G)$-modules. Notice that $R_0\big(D^\omega(G)\big)$ is in fact already generated by these elements as a vector space, since tensor products of modules are taken over the field k. The *Grothendieck ring* $\mathcal{R}\big(D^\omega(G)\big)$ is defined to be the quotient of $R\big(D^\omega(G)\big)$ by $R_0\big(D^\omega(G)\big)$. This $\mathbf{C}$-algebra has a basis consisting of the images of the irreducible $D^\omega(G)$-modules.

The quasi-Hopf algebra $D^\omega(G)$ is quasitriangular [4, 10], with

$$R=\sum_{g,h\in G}\delta_g\bar{1}\otimes\delta_h\bar{g}\quad\text{and}\quad R^{-1}=\sum_{g,h\in G}\theta_{ghg^{-1}}(g,g^{-1})^{-1}\delta_g\bar{1}\otimes\delta_h\overline{g^{-1}}.$$

That is, $R\Delta(a)R^{-1}=\sigma\big(\Delta(a)\big)$ for all $a\in D^\omega(G)$, where σ is the twist automorphism interchanging two factors. If U and V are $D^\omega(G)$-modules, this equation yields an isomorphism between the modules $U\otimes V$ and $V\otimes U$, given by the twist σ followed by the natural action of R. Thus the representation ring $R\big(D^\omega(G)\big)$ is commutative.

We note that if $\beta\colon G\times G\longrightarrow k^\times$ is a normalized 2-cochain, and $\delta\beta$ its coboundary

$$\delta\beta(a,b,c)=\beta(b,c)\beta(a,bc)\beta(ab,c)^{-1}\beta(a,b)^{-1},$$

then $D^{\omega\delta\beta}(G)$ is isomorphic to $D^\omega(G)$ as an algebra. The map $\iota\colon D^\omega(G)\longrightarrow D^{\omega\delta\beta}(G)$ defined by

$$\iota(\delta_g\bar{x})=\frac{\beta(g,x)}{\beta(x,g^x)}\delta_g\bar{x}$$

is an algebra isomorphism. As a quasi-Hopf algebra, $D^{\omega\delta\beta}(G)$ is obtained from $D^\omega(G)$ by the twist element

$$F=\sum_{g,h\in G}\beta(g,h)^{-1}\delta_g\bar{1}\otimes\delta_h\bar{1}.$$

That is, $\Delta\big(\iota(a)\big) = F\Big(\iota \otimes \iota\big(\Delta(a)\big)\Big)F^{-1}$ for all $a \in D^{\omega}(G)$ [4, 7]. The map $\iota: D^{\omega}(G) \longrightarrow D^{\omega\delta\beta}(G)$ induces an isomorphism of representation rings

$$\iota^*: R\big(D^{\omega\delta\beta}(G)\big) \xrightarrow{\sim} R\big(D^{\omega}(G)\big).$$

We point out that $H^3(G, k^\times)$ is finite of order dividing $|G|_{p'}$ (that part of $|G|$ not divisible by the characteristic p of k), and that any 3-cocycle ω is cohomologous to a 3-cocycle of finite order dividing $|G|_{p'}$. This can be shown by an argument exactly analogous to that for 2-cocycles in [6, Lemma 11.38]. In addition, a 3-cocycle of finite order cohomologous to ω may be chosen to be normalized as well. We will assume from now on that ω is normalized and of finite order dividing $|G|_{p'}$.

If x commutes with g and h, then

$$\theta_x(g,h) = \gamma_x(g,h) = \frac{\omega(x,g,h)\omega(g,h,x)}{\omega(g,x,h)},$$

where these are the functions arising in the coproduct and product of $D^{\omega}(G)$ defined above. A calculation shows that this function $\theta_x: C(x) \times C(x) \longrightarrow k^\times$ is a 2-cocycle; that is $\theta_x(ab,c)\theta_x(a,b) = \theta_x(a,bc)\theta_x(b,c)$ for all $a, b, c \in C(x)$.

As an algebra, $D^{\omega}(G)$ is semisimple if and only if the characteristic p of k does not divide the order of G; we outline an argument here. If p does not divide $|G|$, then both kG and $(kG)^*$ are semisimple, and Maschke's Theorem implies that $D^{\omega}(G)$ is semisimple (see [13]). On the other hand, $D^{\omega}(G)/\ker(\epsilon)$ and kt, where $t = \sum_{g\in G}\delta_1\bar{g}$, are both isomorphic to the trivial module k as $D^{\omega}(G)$-modules. If $D^{\omega}(G)$ is semisimple, then k occurs only once in a decomposition of the right regular module $D^{\omega}(G)$ into a direct sum of irreducible $D^{\omega}(G)$-modules. In particular, any composition series of $D^{\omega}(G)$ contains exactly one factor isomorphic with k. This implies that p does not divide $|G|$, as otherwise we have $t \in \ker(\epsilon)$.

In the case $k = \mathbf{C}$, a description of the irreducible $D^{\omega}(G)$-modules is given in [7]; in case ω is trivial, a description of the indecomposable $D^{\omega}(G)$-modules over an arbitrary field k is given in [14]. These results are easily extended. The irreducible (respectively, indecomposable) $D^{\omega}(G)$-modules are indexed by pairs (U, g) where g is a representative of a conjugacy class of G and U is an irreducible (respectively, indecomposable) $k^{\theta_g}C(g)$-module. Here $k^{\theta_g}C(g)$ denotes the twisted group algebra with basis $\{x_h\}_{h\in C(g)}$ and product $x_h x_\ell = \theta_g(h,\ell)x_{h\ell}$. This characterization of $D^{\omega}(G)$-modules arises from an equivalence of categories, which we describe next, following the approach in [14]. It may be helpful to notice first that any $D^{\omega}(G)$-module U is a direct sum of subspaces $U = \sum_{x\in G} U\cdot\delta_x$, and the invertible elements $\bar{g}$ of $D^{\omega}(G)$ permute these subspaces, as $\delta_x\bar{g} = \bar{g}\delta_{g^{-1}xg}$.

For each $x \in G$, there are subspaces $D^{\omega}(x) = \sum_{g\in G} k\delta_x\bar{g}$ and $S^{\omega}(x) = \sum_{g\in C(x)} k\delta_x\bar{g}$ of $D^{\omega}(G)$. Note that $S^{\omega}(x)$ is a subalgebra, with identity element δ_x, isomorphic to the twisted group algebra $k^{\theta_x}C(x)$. Accordingly, we identify these two algebras in the following lemma. If K is a conjugacy class of elements of G, let $D^{\omega}(K) = \sum_{x\in K} D^{\omega}(x)$. Then $D^{\omega}(G) \simeq \oplus_K D^{\omega}(K)$ is a direct sum of two-sided ideals, where K ranges over all conjugacy classes of G. Therefore an indecomposable $D^\omega(G)$-module is a module for precisely one of the $D^\omega(K)$. The irreducible (indecomposable) $D^\omega(G)$-modules are thus characterized by the equivalence of categories given in the following lemma, whose proof is straightforward. We point out that $D^\omega(x)$ is free as a left $S^\omega(x)$-module, so the equivalence of categories in the lemma preserves short exact sequences.

LEMMA 1.1. *Let K be a conjugacy class of G, $x \in K$, and* Mod-$k^{\theta_x}C(x)$ *and* Mod-$D^\omega(K)$ *the categories of right $k^{\theta_x}C(x)$- and $D^\omega(K)$-modules, respectively. Let $U \in$* Mod-$k^{\theta_x}C(x)$ *and $V \in$* Mod-$D^\omega(K)$. *The maps*

$$U \longmapsto U \otimes_{S^\omega(x)} D^\omega(x) \quad \textit{and} \quad V \longmapsto V \cdot \delta_x$$

define an equivalence of categories between Mod-$k^{\theta_x}C(x)$ *and* Mod-$D^\omega(K)$, *where $D^\omega(K)$ acts on $U \otimes_{S^\omega(x)} D^\omega(x)$ by right multiplication in the second factor.*

2. **The case $k = \mathbf{C}$.** In this section, we present results about the representation ring $R\big(D^\omega(G)\big)$ in the special case $k = \mathbf{C}$. These results will be generalized to an arbitrary algebraically closed field in Section 4.

Certain characters of $D^\omega(G)$ are introduced in [1]; these are the functions from $G \times G$ to $\mathbf{C}$ defined for each $D^\omega(G)$-module U by sending a pair (h, g) to the trace $\mathrm{Tr}(\delta_h\bar{g}, U)$ of the action of $\delta_h\bar{g}$ on U. With these characters, we will build certain functions from $R\big(D^\omega(G)\big)$ to $\mathbf{C}$ which are in fact algebra homomorphisms, or characters of the representation ring.

We first define, for each $g \in G$, a linear function $f_g\colon R\big(D^\omega(G)\big) \longrightarrow \mathbf{C}^{\theta_g}C(g)$ by

$$f_g(U) = \sum_{h\in C(g)} \mathrm{Tr}(\delta_h\bar{g}, U)x_h$$

for any $D^\omega(G)$-module U. We will see that f_g is an algebra homomorphism with image the centre $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$ of $\mathbf{C}^{\theta_g}C(g)$. In fact, the product of all f_g, taken over a set of representatives g of conjugacy classes of G, provides an algebra isomorphism from $R\big(D^\omega(G)\big)$ to $\prod_g Z\big(\mathbf{C}^{\theta_g}C(g)\big)$, as is proved in Theorem 2.2 below.

LEMMA 2.1. *The function $f_g\colon R\big(D^\omega(G)\big) \longrightarrow \mathbf{C}^{\theta_g}C(g)$ is an algebra homomorphism.*

PROOF. It may be checked that f_g takes the trivial $D^\omega(G)$-module $\mathbf{C}$ to the identity x_1 of $\mathbf{C}^{\theta_g}C(g)$. Now let U and V be $D^\omega(G)$-modules. Then

$$\begin{aligned} f_g(U)f_g(V) &= \sum_{h,\ell\in C(g)} \theta_g(h,\ell)\,\mathrm{Tr}(\delta_h\bar{g}, U)\,\mathrm{Tr}(\delta_\ell\bar{g}, V)x_{h\ell} \\ &= \sum_{h,\ell\in C(g)} \theta_g(h, h^{-1}\ell)\,\mathrm{Tr}(\delta_h\bar{g}, U)\,\mathrm{Tr}(\delta_{h^{-1}\ell}\bar{g}, V)x_\ell, \end{aligned}$$

where in the second sum, ℓ has been replaced by $h^{-1}\ell$.

On the other hand,

$$f_g(U \otimes V) = \sum_{\ell\in C(g)} \mathrm{Tr}(\delta_\ell\bar{g}, U \otimes V)x_\ell.$$

The action of $\delta_\ell \bar{g}$ on $U \otimes V$ is given by $\Delta(\delta_\ell \bar{g}) = \sum_{h \in G} \gamma_g(h, h^{-1}\ell)\delta_h \bar{g} \otimes \delta_{h^{-1}\ell}\bar{g}$. Therefore

$$\begin{aligned} f_g(U \otimes V) &= \sum_{\ell \in C(g), h \in G} \gamma_g(h, h^{-1}\ell)\,\mathrm{Tr}(\delta_h \bar{g}, U)\,\mathrm{Tr}(\delta_{h^{-1}\ell}\bar{g}, V)x_\ell \\ &= \sum_{h, \ell \in C(g)} \theta_g(h, h^{-1}\ell)\,\mathrm{Tr}(\delta_h \bar{g}, U)\,\mathrm{Tr}(\delta_{h^{-1}\ell}\bar{g}, V)x_\ell \\ &\quad + \sum_{\ell \in C(g), h \in G - C(g)} \gamma_g(h, h^{-1}\ell)\,\mathrm{Tr}(\delta_h \bar{g}, U)\,\mathrm{Tr}(\delta_{h^{-1}\ell}\bar{g}, V)x_\ell, \end{aligned}$$

as θ_g is equal to γ_g on $C(g) \times C(g)$. Comparing the two calculations, we need only see that the second sum above is equal to zero. But when $h \notin C(g)$, the element $\delta_h \bar{g}$ is nilpotent, so its trace on any module is zero. ■

We next show that the image of f_g is contained in $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$. We first give a basis of $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$. We say that $h \in C(g)$ is *θ_g-regular* if $\theta_g(h, c) = \theta_g(c, h)$ for all $c \in C_{C(g)}(h)$. It may be checked that there is a basis of $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$ indexed by conjugacy classes of θ_g-regular elements h of $C(g)$. Each basis element is a sum $\sum a_{h^y} x_{h^y}$ over all elements h^y in the $C(g)$-conjugacy class of h, with coefficients satisfying $a_{h^y} = \theta_g(h, y)/\theta_g(y, h^y)$.

Letting $h, y \in C(g)$, note that

$$\delta_{h^y}\bar{g} = \bar{y}^{-1}\delta_h \bar{y}\bar{g} = \bar{y}^{-1}\left(\frac{\theta_h(y, g)}{\theta_h(g, y)}\delta_h \bar{g}\right)\bar{y}.$$

If U is any $D^\omega(G)$-module then, we have

$$\mathrm{Tr}(\delta_{h^y}\bar{g}, U) = \frac{\theta_h(y, g)}{\theta_h(g, y)}\,\mathrm{Tr}(\delta_h \bar{g}, U) = \frac{\theta_g(h, y)}{\theta_g(y, h^y)}\,\mathrm{Tr}(\delta_h \bar{g}, U).$$

The second equality follows from the definition of θ, as $h, y \in C(g)$. The 2-cocycle property of θ_g on $C(g)$ yields

$$\frac{\theta_g(h, cy)}{\theta_g(cy, h^{cy})} = \frac{\theta_g(h, c)\theta_g(h, y)}{\theta_g(c, h)\theta_g(y, h^y)}$$

whenever $c \in C_{C(g)}(h)$. These equations imply that $\mathrm{Tr}(\delta_{h^y}\bar{g}, U) = 0$ if h is not θ_g-regular, and that $f_g(U)$ is in $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$.

Next we will need to consider characters of a twisted group algebra $\mathbf{C}^\alpha H$, where α is a 2-cocycle on a finite group H. For facts about characters of finite dimensional algebras, we refer to [6, Section 9B], and note that $\mathbf{C}^\alpha H$ is semisimple [11]. Specifically, a character is a function from $\mathbf{C}^\alpha H$ to $\mathbf{C}$ given by the trace function on a $\mathbf{C}^\alpha H$-module. These functions are not algebra homomorphisms in general. The character of an arbitrary $\mathbf{C}^\alpha H$-module is the sum of the characters of its irreducible direct summands. The irreducible characters, that is characters corresponding to irreducible modules, are linearly independent over $\mathbf{C}$.

We will consider the vector space generated by $\mathbf{C}^\alpha H$-modules with direct sum for addition. By the trace function of an element of this space, we will mean the corresponding linear combination of trace functions of $\mathbf{C}^\alpha H$-modules.

THEOREM 2.2. *The product π of the maps f_g induces an isomorphism of algebras*

$$R\big(D^{\omega}(G)\big) \xrightarrow{\sim} \prod_{g} Z\big(\mathbf{C}^{\theta_g}C(g)\big),$$

the product taken over a set of representatives g of conjugacy classes of G. In particular, the representation ring $R\big(D^{\omega}(G)\big)$ is semisimple.

PROOF. Lemma 2.1 and the discussion following it show that each f_g is an algebra homomorphism with image contained in $Z\big(\mathbf{C}^{\theta_g}C(g)\big)$. It remains to prove that π is a bijection. Let $a \in R\big(D^{\omega}(G)\big)$ with $\pi(a) = 0$. Fix $x \in G$ and let $g \in C(x)$. Then $f_g(a) = 0$ implies that $\operatorname{Tr}(\delta_x\bar{g}, a) = 0$, where we have extended the trace function linearly to $R\big(D^{\omega}(G)\big)$. But $\operatorname{Tr}(\delta_x\bar{g}, a) = \operatorname{Tr}(\delta_x\bar{g}, a\delta_x)$, where here we may consider $a\delta_x$ as an element of the vector space generated by $\mathbf{C}^{\theta_x}C(x)$-modules via the equivalence of categories given in Lemma 1.1. In other words, we have $\operatorname{Tr}(x_g, a\delta_x) = 0$ for all $g \in C(x)$. As the irreducible characters of $\mathbf{C}^{\theta_x}C(x)$ are linearly independent, $a\delta_x = 0$ in the vector space generated by $\mathbf{C}^{\theta_x}C(x)$-modules. But this is true for all $x \in G$. As $D^{\omega}(G)$-modules are determined by their x-components, we have $a = 0$. Thus we have shown that π is injective.

To complete the proof, we must show that π is also surjective, which will follow once we see that the dimensions of $R\big(D^{\omega}(G)\big)$ and $\prod_g Z\big(\mathbf{C}^{\theta_g}C(g)\big)$ are the same. By Lemma 1.1, the dimension of $R\big(D^{\omega}(G)\big)$ is equal to the sum, over a set of representatives g of conjugacy classes of G, of the number of irreducible $\mathbf{C}^{\theta_g}C(g)$-modules. This is also the dimension of $\prod_g Z\big(\mathbf{C}^{\theta_g}C(g)\big)$. ∎

Now we may write down a table consisting of the values of the complete set of characters of the semisimple algebra $R\big(D^{\omega}(G)\big)$ on the irreducible $D^{\omega}(G)$-modules. Such a character maps a $D^{\omega}(G)$-module U to

$$\frac{1}{\deg\rho}\sum_{h\in C(g)} \operatorname{Tr}(\delta_h\bar{g}, U)\rho(x_h),$$

where $g \in G$ and ρ is an irreducible character of $\mathbf{C}^{\theta_g}C(g)$; that is, ρ is the trace function of an irreducible $\mathbf{C}^{\theta_g}C(g)$-module. This table is analogous to the character table of a finite group.

3. **Brauer characters.** Here we present an analog of Brauer characters for a twisted group algebra, and some results about their connection to Brauer characters of a representation group.

Let H be a finite group, k an algebraically closed field of characteristic p, and $\alpha\colon H \times H \longrightarrow k^{\times}$ a 2-cocycle. As before, we denote by $k^{\alpha}H$ the twisted group algebra that has basis $\{x_h\}_{h\in H}$ with multiplication given by $x_h x_\ell = \alpha(h, \ell)x_{h\ell}$. We will assume that α is normalized so that x_1 is the multiplicative identity of $k^{\alpha}H$, and also that α is of finite order m dividing $|H|_{p'}$. A calculation shows that the elements x_h of $k^{\alpha}H$ all have finite order dividing $m \cdot |H|$. The representations of $k^{\alpha}H$ correspond to projective k-representations of H with associated 2-cocycle α [11, 12].

Let E be a *representation group* for H. That is, there is a central extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\pi} H \longrightarrow 1$$

of H such that any projective k-representation of H may be lifted to an ordinary k-representation of E which on a section of π is projectively equivalent to the original one [6, 12]. We will recall the explicit formula for so lifting a projective representation $\rho_U \colon H \longrightarrow \mathrm{GL}(U)$, with associated 2-cocycle α, to an ordinary representation $\sigma_U \colon E \longrightarrow \mathrm{GL}(U)$ [6, 11]. We will denote by $\mathcal{L}(U)$ the vector space U with the structure of a kE-module via σ_U. Fix a section $\{y_g\}_{g \in H}$ of π with $y_1 = 1$, and let $a \colon H \times H \longrightarrow A$ be the corresponding 2-cocycle provided by multiplication in E. That is, $y_g y_h = a(g,h) y_{gh}$ for all $g, h \in H$. Let $\lambda \in \hat{A}$ be the character such that

$$\lambda\big(a(g,h)\big) = \alpha(g,h)\mu(g)\mu(h)\mu(gh)^{-1}$$

for some function $\mu \colon G \longrightarrow k^{\times}$, and all $g, h \in H$. Then $\sigma_U(a y_g) = \lambda(a)\mu(g)\rho_U(g)$ for all $a \in A, g \in H$. Note that μ and λ depend only on α and a, and not on the given representation.

Another way to construct σ_U is by using the isomorphism $kE \simeq \prod_{\alpha \in T} k^{\alpha}H$, where T is a transversal of the group $B^2(H, k^{\times})$ of 2-coboundaries in the group $Z^2(H, k^{\times})$ of 2-cocycles [12, Theorem 3.3.5]. The explicit map $kE \longrightarrow k^{\alpha}H$ is given by sending $a y_g$ to $\lambda(a)\mu(g)x_g$, where λ and μ are as above. A $k^{\alpha}H$-module may be lifted to kE via this surjection. Morphisms of $k^{\alpha}H$-modules then become morphisms of kE-modules, and short exact sequences are preserved.

Let $r = |E|_{p'}^2$. The r-th roots of unity in k form a cyclic group of order r. For the rest of this section, fix an isomorphism between the group of r-th roots of unity in k and the group of r-th roots of unity in $\mathbf{C}$.

Let U be a $k^{\alpha}H$-module, and g a p-regular element of H; that is, p does not divide the order of g. The element x_g of $k^{\alpha}H$ has finite order dividing $r = |E|_{p'}^2$, so its eigenvalues on U are r-th roots of unity, and may be lifted to $\mathbf{C}$ via the isomorphism chosen above. Define the *Brauer character* of U to be the function from the set of p-regular elements of H to $\mathbf{C}$ given by sending g to the sum of the lifts of the eigenvalues of x_g on U to $\mathbf{C}$. In case α is the trivial 2-cocycle, this Brauer character is the usual one for H. For a given p-regular element $g \in H$, define the function $t_{g,\alpha}$ on $k^{\alpha}H$-modules by setting $t_{g,\alpha}(U)$ equal to the Brauer character of U evaluated at g. If E is a representation group of H, we will denote by $t_y(V)$ the Brauer character of a kE-module V evaluated at a p-regular element y of E.

LEMMA 3.1. *Let U be a $k^{\alpha}H$-module, g a p-regular element of H, and $\mathcal{L}(U)$ the lift of U to a kE-module defined above. Then $t_{y_g}\big(\mathcal{L}(U)\big)$ is a nonzero scalar multiple of $t_{g,\alpha}(U)$. Further, the same nonzero scalar multiple is involved in any such lift, so that $t_{y_g} \circ \mathcal{L}$ is a nonzero scalar multiple of $t_{g,\alpha}$.*

PROOF. It follows from the construction of the module $\mathcal{L}(U)$ that the action of x_g on U and that of y_g on $\mathcal{L}(U)$ differ by the scalar $\mu(g)$. It may be checked that $\mu(g)$ is an r-th root of unity, where $r = |E|_{p'}^2$. Lifting eigenvalues to $\mathbf{C}$ then, and summing, we obtain the desired result. ■

We will need to exploit the connection between induced kE-modules and induced $k^\alpha H$-modules. Let L be a subgroup of H, and consider α also to be a 2-cocycle of L by restriction. If U is a $k^\alpha L$-module, then U may be lifted to an ordinary $k\pi^{-1}(L)$-module $\mathcal{L}(U)$ using the same procedure as for H and E (see [12, Lemma 3.3.1]).

LEMMA 3.2. *Let U be a $k^\alpha L$-module and $U \otimes_{k^\alpha L} k^\alpha H$ the corresponding induced $k^\alpha H$-module. Then $\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha H)$ is isomorphic to the induced kE-module $\mathcal{L}(U) \otimes_{kJ} kE$, where $J = \pi^{-1}(L)$.*

PROOF. Let $g_1, \dots, g_t$ be a transversal of L in H with $g_1 = 1$. Then $y_{g_1}, \dots, y_{g_t}$ is a transversal of J in E. Define a linear map from $\mathcal{L}(U) \otimes_{kJ} kE$ to $\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha H)$ by

$$u \otimes y_{g_i} \longmapsto \mu(g_i) u \otimes x_{g_i},$$

where we have identified the underlying vector spaces of U and $\mathcal{L}(U)$. It may be checked that this map is an isomorphism of kE-modules. ■

Let g be a p-regular element of H, and consider the function $t_{g,\alpha}$ on $k^\alpha\langle g\rangle$-modules, by replacing H by $\langle g\rangle$ in the above development. We may extend $t_{g,\alpha}$ linearly to the $\mathbf{C}$-vector space generated by $k^\alpha\langle g\rangle$-modules (with direct sum for addition). In the next section, we will use $t_{g,\alpha}$ to define a function generalizing f_g from Section 2. The following lemma will be used in the proof that such a function is an algebra homomorphism.

LEMMA 3.3. *Let g be a p-regular element of H, and L a proper subgroup of $\langle g\rangle$. Then the kernel of $t_{g,\alpha}$ contains any $k^\alpha\langle g\rangle$-module induced from $k^\alpha L$.*

PROOF. Let E be a representation group for $\langle g\rangle$. By Lemma 3.1, $t_{y_g} \circ \mathcal{L}$ is a nonzero scalar multiple of $t_{g,\alpha}$. Therefore it suffices to show that $t_{y_g}\bigl(\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha\langle g\rangle)\bigr) = 0$ for any $k^\alpha L$-module U. By Lemma 3.2, $\mathcal{L}(U \otimes_{k^\alpha L} k^\alpha\langle g\rangle) \simeq \mathcal{L}(U) \otimes_{kJ} kE$ as kE-modules, where $J = \pi^{-1}(L)$ is a proper subgroup of E containing A. But t_{y_g} factors through $\langle y_g\rangle \leq E$; that is $t_{y_g} = \tilde{t}_{y_g} \circ \downarrow^E_{\langle y_g\rangle}$, where $\tilde{t}_{y_g}$ is the corresponding trace function on $k\langle y_g\rangle$-modules, and $\downarrow^E_{\langle y_g\rangle}$ denotes restriction of modules [3]. Using $\uparrow^E_J$ to denote induction of modules, we apply $\downarrow^E_{\langle y_g\rangle}$ to $\mathcal{L}(U) \uparrow^E_J = \mathcal{L}(U) \otimes_{kJ} kE$, and the Mackey Subgroup Theorem yields

$$\mathcal{L}(U) \uparrow^E_J \downarrow^E_{\langle y_g\rangle} \simeq \sum_{\sigma \in J\backslash E/\langle y_g\rangle} \mathcal{L}(U)^\sigma \downarrow^{J^\sigma}_{J^\sigma \cap \langle y_g\rangle} \uparrow^{\langle y_g\rangle}_{J^\sigma \cap \langle y_g\rangle} .$$

Now, as E is generated by y_g and A, and J^σ is a proper subgroup of E containing A, we have $y_g \notin J^\sigma$. Therefore $J^\sigma \cap \langle y_g\rangle$ is a proper subgroup of $\langle y_g\rangle$. Thus $\mathcal{L}(U) \uparrow^E_J \downarrow^E_{\langle y_g\rangle}$ is in the image of induction from proper subgroups of $\langle y_g\rangle$, and so is in the kernel of $\tilde{t}_{y_g}$ [3]. ■

We return to the situation where ω is a 3-cocycle of a finite group G, of finite order dividing $|G|_{p'}$. The next lemma deals with a certain direct summand of the tensor product $U \otimes V$ of two $D^\omega(G)$-modules U and V.

LEMMA 3.4. *Let U and V be $D^\omega(G)$-modules, $\ell \in G$, and $g \in C(\ell)$. Consider $(U \otimes V)\delta_\ell$ to be a $k^{\theta_\ell}\langle g\rangle$-module via Lemma 1.1 and restriction. Then the subspace*

$$\sum_{h \in G - C(g)} U\delta_h \otimes V\delta_{h^{-1}\ell}$$

of $(U \otimes V)\delta_\ell$ is a $k^{\theta_\ell}\langle g\rangle$-submodule, and is a sum of modules induced from proper subgroups of $\langle g\rangle$.

PROOF. First note that the given subspace is indeed a $k^{\theta_\ell}\langle g\rangle$-submodule, as the sum is over elements h not in $C(g)$, and $\delta_h\bar{g} = \bar{g}\delta_{g^{-1}hg}$. We may assume that U and V are indecomposable $D^\omega(G)$-modules. By Lemma 1.1 then, there are elements $x, y \in G$ such that

$$U \simeq U\delta_x \otimes_{S^\omega(x)} D^\omega(x) \quad \text{and} \quad V \simeq V\delta_y \otimes_{S^\omega(y)} D^\omega(x).$$

Consider a $k^{\theta_\ell}\langle g\rangle$-submodule

$$M = \sum_{(x,y)} U\delta_x \otimes V\delta_y$$

of $\sum_{h \in G - C(g)} U\delta_h \otimes V\delta_{h^{-1}\ell}$, where the pair (x, y) ranges over all $\langle g\rangle$-conjugates of $(h_0, h_0^{-1}\ell)$ for a fixed $h_0 \in G - C(g)$. The $\langle g\rangle$-conjugates of $(h_0, h_0^{-1}\ell)$ form an orbit as a $\langle g\rangle$-set which is isomorphic to $L \setminus \langle g\rangle$ for a proper subgroup L of $\langle g\rangle$, as h_0 is not in $C(g)$. It may be checked that M is induced from the $k^{\theta_\ell}L$-module $U\delta_{h_0} \otimes V\delta_{h_0^{-1}\ell}$; the map from $(U\delta_{h_0} \otimes V\delta_{h_0^{-1}\ell}) \otimes_{k^{\theta_\ell}L} k^{\theta_\ell}\langle g\rangle$ to M defined by

$$(u\delta_{h_0} \otimes v\delta_{h_0^{-1}\ell}) \otimes \delta_\ell\overline{g^j} \longmapsto u\delta_{h_0}\overline{g^j} \otimes v\delta_{h_0^{-1}\ell}\overline{g^j}\gamma_{g^j}(h_0, h_0^{-1}\ell)$$

is a $k^{\theta_\ell}\langle g\rangle$-module isomorphism. ∎

4. **Characters of the Grothendieck ring.** In this section, we will give a generalization of Theorem 2.2 to an arbitrary algebraically closed field k of characteristic p. Let r be the square of the least common multiple of all $|E|_{p'}$, where E ranges over a set of representation groups for the subgroups of G. For the rest of this section, fix an isomorphism between the r-th roots of unity in k and in $\mathbf{C}$.

Given $g \in G$, the values of θ_g are $|G|_{p'}$-th roots of unity in k, by the restriction on the order of ω. Therefore, they may be lifted to $\mathbf{C}$ by our chosen isomorphism of r-th roots of unity in k and in $\mathbf{C}$. Denote by $\Theta_g \colon C(g) \times C(g) \longrightarrow \mathbf{C}^\times$ the lift of the map θ_g with respect to this isomorphism, and note that Θ_g is a 2-cocycle on $C(g)$ as well, but with values in $\mathbf{C}^\times$. Thus we may form the twisted group algebra $\mathbf{C}^{\Theta_g}C(g)$.

Now let g be a p-regular element of G. Define a linear function $f_g \colon R\big(D^\omega(G)\big) \longrightarrow \mathbf{C}^{\Theta_g}C(g)$ by

$$f_g(U) = \sum_{h \in C(g)} t_{g,\theta_h}(U\delta_h)x_h,$$

where $U\delta_h$ is considered to be a $k^{\theta_h}C(h)$-module via Lemma 1.1, and the functions t_{g,θ_h} are defined in Section 3. We will show that f_g is an algebra homomorphism and that its image is contained in $Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$.

We point out that the proofs of results in Section 2 cannot be translated directly in this situation. The functions t_{g,θ_h} require looking at eigenvalues of invertible operators. By contrast, the proof of Lemma 2.1 depended on considering traces of nilpotent operators.

THEOREM 4.1. *The function* $f_g\colon R\big(D^\omega(G)\big) \to \mathbf{C}^{\Theta_g}C(g)$ *is an algebra homomorphism.*

PROOF. It may be checked that f_g takes the trivial $D^\omega(G)$-module k to the identity x_1 of $\mathbf{C}^{\Theta_g}C(g)$. Now let U and V be $D^\omega(G)$-modules. Then

$$\begin{aligned} f_g(U)f_g(V) &= \sum_{h,\ell\in C(g)} \Theta_g(h,\ell)t_{g,\theta_h}(U\delta_h)t_{g,\theta_\ell}(V\delta_\ell)x_{h\ell} \\ &= \sum_{h,\ell\in C(g)} \Theta_g(h,h^{-1}\ell)t_{g,\theta_h}(U\delta_h)t_{g,\theta_{h^{-1}\ell}}(V\delta_{h^{-1}\ell})x_\ell, \end{aligned}$$

where in the second sum, we have replaced ℓ by $h^{-1}\ell$. On the other hand,

$$\begin{aligned} f_g(U\otimes V) &= \sum_{\ell\in C(g)} t_{g,\theta_\ell}\big((U\otimes V)\delta_\ell\big)x_\ell \\ &= \sum_{\ell\in C(g)} t_{g,\theta_\ell}\Big(\sum_{h\in G} U\delta_h\otimes V\delta_{h^{-1}\ell}\Big)x_\ell. \end{aligned}$$

Now t_{g,θ_ℓ} is the sum of lifts of eigenvalues of $\delta_\ell\bar{g}$ on a module to $\mathbf{C}$, and the action of $\delta_\ell\bar{g}$ on a tensor product is given by

$$\Delta(\delta_\ell\bar{g}) = \sum_{h\in G}\gamma_g(h,h^{-1}\ell)\delta_h\bar{g}\otimes\delta_{h^{-1}\ell}\bar{g}.$$

Note that when $h\in C(g)$, $U\delta_h\otimes V\delta_{h^{-1}\ell}$ is a $k^{\theta_\ell}\langle g\rangle$-submodule of $(U\otimes V)\delta_\ell$, and $\gamma_g(h,h^{-1}\ell)$ lifts to $\Theta_g(h,h^{-1}\ell)$ in $\mathbf{C}$, since $\gamma_g=\theta_g$ on $C(g)\times C(g)$. Therefore

$$\begin{aligned} f_g(U\otimes V) = {}&\sum_{\ell,h\in C(g)} \Theta_g(h,h^{-1}\ell)t_{g,\theta_h}(U\delta_h)t_{g,\theta_{h^{-1}\ell}}(V\delta_{h^{-1}\ell})x_\ell \\ &+\sum_{\ell\in C(g)} t_{g,\theta_\ell}\Big(\sum_{h\in G-C(g)} U\delta_h\otimes V\delta_{h^{-1}\ell}\Big)x_\ell. \end{aligned}$$

By Lemmas 3.3 and 3.4, the second summand above is zero, and so $f_g(U\otimes V)$ is equal to $f_g(U)f_g(V)$. ■

For the next lemma, we recall the description of $Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$ given in Section 2. It has a basis indexed by Θ_g-regular elements h, each basis element being a sum $\sum a_{h^y}x_{h^y}$ over all elements h^y in the $C(g)$-conjugacy class of h, with coefficients satisfying $a_{h^y} = \Theta_g(h,y)/\Theta_g(y,h^y)$. We note that by the construction of Θ_g, h is Θ_g-regular if and only if h is θ_g-regular.

LEMMA 4.2. *The image of* f_g *is contained in* $Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$.

PROOF. Let U be a $D^{\omega}(G)$-module and $h, y \in C(g)$. As $\delta_{h^y}\bar{g} = \bar{y}^{-1}(\frac{\theta_h(y,g)}{\theta_h(g,y)}\delta_h\bar{g})\bar{y}$, the action of $\frac{\theta_h(y,g)}{\theta_h(g,y)}\delta_h\bar{g}$ on $U\delta_h$ corresponds to the action of $\delta_{h^y}\bar{g}$ on $U\delta_{h^y}$ under the vector space isomorphism sending $U\delta_h$ to $U\delta_{h^y}$ defined by right action by $\bar{y}$. Thus the eigenvalues of $\delta_{h^y}\bar{g}$ on $U\delta_{h^y}$ are $\frac{\theta_h(y,g)}{\theta_h(g,y)}$ times the eigenvalues of $\delta_h\bar{g}$ on $U\delta_h$. So we have

$$t_{g,\theta_{h^y}}(U\delta_{h^y}) = \frac{\Theta_h(y,g)}{\Theta_h(g,y)} t_{g,\theta_h}(U\delta_h).$$

If h is Θ_g-regular, this is what we needed to show, as $\frac{\Theta_h(y,g)}{\Theta_h(g,y)} = \frac{\Theta_g(h,y)}{\Theta_g(y,h^y)}$ by the definitions. If h is not Θ_g-regular, these equations force $t_{g,\theta_h}(U\delta_h) = 0$ (see Section 2). ∎

Finally, we will consider the Grothendieck ring $\mathcal{R}(D^{\omega}(G))$, which is the quotient of $R(D^{\omega}(G))$ by the ideal $R_0(D^{\omega}(G))$ of short exact sequences defined in Section 1.

LEMMA 4.3. *The ideal* $R_0(D^{\omega}(G))$ *is the kernel of the homomorphism*

$$\pi : R(D^{\omega}(G)) \longrightarrow \prod_g Z(\mathbb{C}^{\Theta_g}C(g))$$

given by the product of the maps f_g *taken over a set of representatives* g *of conjugacy classes of* p*-regular elements of* G.

PROOF. First let $0 \to U' \to U \to U'' \to 0$ be a short exact sequence of $D^{\omega}(G)$-modules. Now, $U' = \sum_{x \in G} U'\delta_x$ as a direct sum of vector spaces, and similarly for the others. The $D^{\omega}(G)$-module maps induce linear maps $U'\delta_x \to U\delta_x \to U''\delta_x$, which correspond to maps of $k^{\theta_x}C(x)$-modules by Lemma 1.1. That is, for each $x \in G$, we have a short exact sequence of $k^{\theta_x}C(x)$-modules

$$0 \longrightarrow U'\delta_x \longrightarrow U\delta_x \longrightarrow U''\delta_x \longrightarrow 0.$$

Fix $x \in G$ and let g be a p-regular element of $C(x)$. Let E be a representation group of $C(x)$, with $E/A \simeq C(x)$ for a central subgroup A of E. We will use notation as in Section 3. We claim that $t_{g,\theta_x}(U\delta_x - U'\delta_x - U''\delta_x) = 0$. Now, the above short exact sequence of $k^{\theta_x}C(x)$-modules may be lifted, by the discussion at the beginning of Section 3, to a short exact sequence $\mathcal{L}(U\delta_x - U'\delta_x - U''\delta_x)$ of kE-modules. By Lemma 3.1, $t_{y_g} \circ \mathcal{L}$ is a nonzero scalar multiple of t_{g,θ_x}. Any short exact sequence is in the kernel of t_{y_g} [3], so we do have $t_{g,\theta_x}(U\delta_x - U'\delta_x - U''\delta_x) = 0$. Therefore, $f_g(U - U' - U'') = 0$ for all p-regular elements g of G, and so $R_0(D^{\omega}(G))$ is contained in the kernel of π.

Now let $b \in R(D^{\omega}(G))$ with $\pi(b) = 0$. Fix $x \in G$ and let g be a p-regular element of $C(x)$. Then $f_g(b) = 0$ implies that $t_{g,\theta_x}(b\delta_x) = 0$ for all $x \in C(g)$, where $b\delta_x$ is considered to be an element of the vector space generated by $k^{\theta_x}C(x)$-modules. Lift $b\delta_x$ to E; that is, lift all modules involved in $b\delta_x$ to E and form the corresponding linear combination $\mathcal{L}(b\delta_x)$. By Lemma 3.1, $\mathcal{L}(b\delta_x)$ is in the kernel of t_{y_g} for all p-regular elements g of $C(x)$. Therefore $\mathcal{L}(b\delta_x)$ is in the kernel of t_{ay_g} for all elements ay_g of E with g a p-regular element and $a \in A$, as a simply acts as the scalar $\lambda(a)\mu(1)$ on any such module lifted from $C(x)$. But all p-regular elements of E are of this form. This implies that $\mathcal{L}(b\delta_x)$ is in the ideal of short exact sequences of kE-modules [2]. The short exact sequences which appear in $\mathcal{L}(b\delta_x)$ involve only kE-modules which correspond to $k^{\theta_x}C(x)$-modules. Thus $b\delta_x$ is in the vector space generated by short exact sequences of $k^{\theta_x}C(x)$-modules, and this is true for all $x \in G$. By the characterization of $D^{\omega}(G)$-modules discussed in Section 1, each is determined by its x-components, where x ranges over a set of representatives of conjugacy classes of G. And a short exact sequence of $k^{\theta_x}C(x)$-modules corresponds to a short exact sequence of $D^{\omega}(G)$-modules, as noted in the text preceding Lemma 1.1. Thus $b \in R_0\big(D^{\omega}(G)\big)$. ∎

THEOREM 4.4. *The product π of the maps f_g induces an algebra isomorphism*

$$\mathcal{R}\big(D^{\omega}(G)\big) \xrightarrow{\sim} \prod_g Z\big(\mathbf{C}^{\Theta_g}C(g)\big),$$

the product taken over a set of representatives g of conjugacy classes of p-regular elements of G. In particular, the Grothendieck ring $\mathcal{R}\big(D^{\omega}(G)\big)$ is semisimple.

PROOF. By Lemma 4.3, the homomorphism π induces an injection from $\mathcal{R}\big(D^{\omega}(G)\big)$ to $\prod_g Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$. It remains to prove that π is a surjection, which will follow once we see that the dimensions of these two finite dimensional algebras are the same.

The dimension of the Grothendieck ring $\mathcal{R}\big(D^{\omega}(G)\big)$ is equal to the number of irreducible $D^{\omega}(G)$-modules. By the characterization given in Lemma 1.1, this is

$$\sum_x \text{(number of irreducible } k^{\theta_x}C(x)\text{-modules)},$$

where the sum is taken over a set of representatives x of conjugacy classes in G. But the number of irreducible $k^{\theta_x}C(x)$-modules is the number of conjugacy classes in $C(x)$ of p-regular θ_x-regular elements [5, 12].

On the other hand, the dimension of $\prod_g Z\big(\mathbf{C}^{\Theta_g}C(g)\big)$ is equal to

$$\sum_g \text{(number of } \Theta_g\text{-regular conjugacy classes in } C(g)\text{)},$$

the sum taken over a set of representatives g of conjugacy classes of p-regular elements in G.

But these quantities are just two different ways of counting the orbits in the G-set

$$\{(x,g) \mid x \in G, g \text{ is a } p\text{-regular element of } G, xg = gx, \text{ and } x \text{ is } \Theta_g\text{-regular}\},$$

where G acts by conjugation on each factor. This follows from the observation that x is a Θ_g-regular element of $C(g)$ if and only if g is a θ_x-regular element of $C(x)$. To see this, note that when g and x commute,

$$\frac{\theta_g(x,c)}{\theta_g(c,x)} = \frac{\theta_x(c,g)}{\theta_x(g,c)}$$

whenever $c \in C(x) \cap C(g)$. ∎

The theorem allows us to write down a table consisting of the values of the complete set of characters of the semisimple ring $\mathcal{R}\big(D^{\omega}(G)\big)$ on the images of the irreducible $D^{\omega}(G)$-modules. Such a character maps the image of a $D^{\omega}(G)$-module U to

$$\frac{1}{\deg \rho} \sum_{h \in C(g)} t_{g,\theta_h}(U\delta_h)\rho(x_h),$$

where $g \in G$ and ρ is an irreducible character of $\mathbf{C}^{\Theta_g}C(g)$. These characters therefore distinguish $D^{\omega}(G)$-modules up to their composition factors, and this table of characters is analogous to the Brauer character table of a finite group.

REFERENCES

1. P. Bantay, *Orbifolds, Hopf algebras, and the moonshine*, Lett. Math. Phys. **22**(1991), 187–194.
2. D. J. Benson, *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*, Cambridge Univ. Press, Cambridge, 1991.
3. D. J. Benson and R. A. Parker, *The Green ring of a finite group*, J. Algebra **87**(1984), 290–331.
4. V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge Univ. Press, 1994.
5. S. B. Conlon, *Twisted group algebras and their representations*, J. Austral. Math. Soc. **4**(1964), 152–173.
6. C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders, Volume I*, Wiley, 1981.
7. R. Dijkgraaf, V. Pasquier, and P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models*, Nuclear Phys. B, Proc. Suppl. **18B**(1990), 60–72.
8. C. Dong and G. Mason, *On the operator content of nilpotent orbifold models*, preprint, 1994.
9. V. G. Drinfel'd, *Quantum groups*. In: Proc. Int. Congr. Math. at Berkeley, Amer. Math. Soc., 1986, 798–820.
10. ———, *Quasi-Hopf algebras*, Leningrad Math. J. **1**(1990), 1419–1457.
11. I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, 1976.
12. G. Karpilovsky, *Projective Representations of Finite Groups*, Marcel Dekker, 1985.
13. M. Lorenz and D. S. Passman, *Two applications of Maschke's Theorem*, Comm. Alg. **8**(1980), 1853–1866.
14. G. Mason, *The quantum double of a finite group and its role in conformal field theory*. In: Groups '93, London Math. Soc. Lecture Notes **212**, Cambridge Univ. Press, 1995, 409–417.
15. S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conference Series in Math. **82**, Amer. Math. Soc., 1993.
16. S. J. Witherspoon, *The Representation Ring of the Quantum Double of a Finite Group*, J. Algebra **179**(1996), 305–329.

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